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Image Refocus in Geometrical Optical Phase Space

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Abstract: We introduce matrix optics and phase space methods as general modeling methods to analyze problems involving 4D light field imaging systems. Specifically we demonstrate the use of these methods for analyzing the image refocus process.

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1. Introduction and Theoretical Background

An emerging trend in imaging systems is to share the image synthesis and reconstruction process between the optics and computation [1,2]. Here we provide matrix optics and phase space methods [3,4] to analyze image refocus [5–7]. We demonstrate the principles of refocus in the spatial and frequency domains [8], within the paraxial approximation.

We choose coordinates so that a light ray (Fig. 1) moves along the general z direction. The position where a ray intersects a plane parallel to the x-y plane is given by the two-dimensional vector $\mathbf{q} = (x, y)^T$ [8,9]. Its direction is given by the direction cosines grouped into the two-dimensional vector $\mathbf{p} = (p_1, p_2)^T$. The location of a ray in phase space is then given by the 4D position vector $\mathbf{u} = (\mathbf{q}, \mathbf{p})^T$. This vector obeys the Hamilton equations, which, making use of the Poisson bracket and defining the Lie operator as $\hat{L}_H = \{\cdot, H\}$, can be written as

$$\frac{d\mathbf{u}}{dz} = \{\mathbf{u}, H\} = \hat{L}_H \mathbf{u}. \qquad (1)$$

Similarly, the phase space density $\rho(\mathbf{q}, \mathbf{p}, z)$, also known as the light field, evolves along a line parallel to the z-axis as $\frac{\partial \rho}{\partial z} = -\hat{L}_H \rho$. The Hamiltonian H does not depend explicitly on z, so the solutions to Eq. (1) and that for the light field can be written as $\mathbf{u}(z) = \exp[(z - z_i)\hat{L}_H]\mathbf{u}_i$ and $\rho(\mathbf{q}, \mathbf{p}, z) = \exp[-(z - z_i)\hat{L}_H]\rho(\mathbf{q}, \mathbf{p}, z_i)$. One possible solution is the free space propagation solution

$$\mathbf{u}(z) = \exp\left[(z - z_i)\mathbf{p}.\partial_{\mathbf{q}}\right]\mathbf{u}_i = \hat{T}(z - z_i)\mathbf{u}_i \qquad \rho(\mathbf{q}, \mathbf{p}, z) = \hat{T}[-(z - z_i)]\rho(\mathbf{q}, \mathbf{p}, z_i). \qquad (2)$$

The general imaging process is the integral projection onto the $\mathbf{q}$-plane, given by

$$I(\mathbf{q}) = \int_{\Omega_{\mathbf{p}}} \rho(\mathbf{q}, \mathbf{p})\, d\mathbf{p}, \qquad (3)$$

where ρ is the phase space at the image plane, $\Omega_{\mathbf{p}}$ is the entire $\mathbf{p}$ space and $I(\mathbf{q})$ is the 2D image.

2. Imaging and Refocus

If we consider a simple optical system, as shown in Fig. 2, that acts on the phase space position vector, the resulting ray-transfer matrix is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{v}{f} & u + v\left(1 - \frac{u}{f}\right) \\ -\frac{1}{f} & 1 - \frac{u}{f} \end{pmatrix} = \begin{pmatrix} -\frac{v}{u} & 0 \\ -\frac{1}{f} & -\frac{u}{v} \end{pmatrix}, \qquad (4)$$

from which we obtain the thin lens equation $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$. From this we show that refocus to a different depth is equivalent to applying the free space propagation operator to the phase space at the image plane, followed by an integral projection onto the $\mathbf{q}$-plane. To focus onto an object at a closer distance, u, we must have a larger distance between the lens and the image plane v. For an image focused at a depth u_{old}, we define the phase space density at the image plane, v_{old} from the lens, as $\rho(q, p)$. From Fig. 2, to refocus at the object u_{new} away, assuming that $v_{new} = v_{old} + \xi$, the refocused image is then

$$I'(\mathbf{q}) = \int_{\Omega_{\mathbf{p}}} \hat{T}(-\xi)\rho(\mathbf{q}, \mathbf{p})d\mathbf{p} = \int_{\Omega_{\mathbf{p}}} \rho(\mathbf{q} - \xi\mathbf{p}, \mathbf{p})d\mathbf{p}. \qquad (5)$$

As can be seen in Eq. (5), the operator $\hat{T}(-\xi)$ performs shears on the light field along the $\mathbf{q}$-plane.

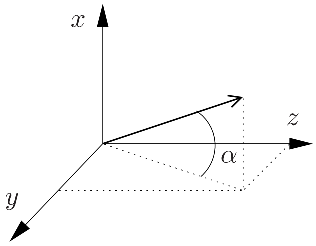

Fig. 1. *Coordinates are chosen so that light rays move in the general z direction. The optical direction cosines of the ray are $p_1 = n\frac{dx}{ds}$ and $p_2 = n\frac{dy}{ds}$, where n is the refractive index, and ds is the infinitesimal length element tangential to the ray. Here, $p_1 = n\frac{dx}{ds} = n\sin\alpha \approx n\alpha$.*

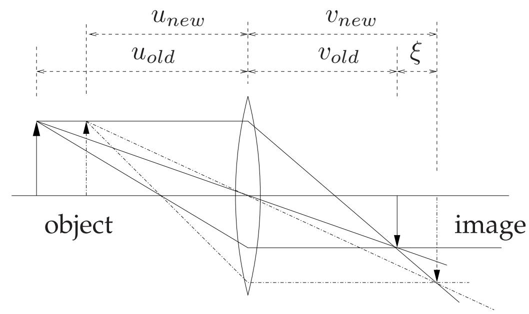

Fig. 2. *Refocus from a farther object to a closer object.*

3. Imaging and Refocus in Spatial Frequency domain

This imaging and refocus process is a slice in the 4D spatial frequency domain. To show this, we introduce the 2D Fourier transform in the $\mathbf{q}$-plane as the operator $\mathcal{F}_{\mathbf{q}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\mathbf{q}\exp(-i\mathbf{k_q}.\mathbf{q})$ and its inverse as $\mathcal{F}_{\mathbf{q}}^{-1}$. The vectors $\mathbf{k_q} = (k_x, k_y)^T$ and $\mathbf{k_p} = (k_{p_1}, k_{p_2})^T$ are the 2D spatial frequency vectors of $\mathbf{q}$ and $\mathbf{p}$ respectively. Here we also use the results for taking the Fourier transform of derivatives, $\mathcal{F}\left(\partial_x^n f(x)\right) = (ik)^n \mathcal{F}\left(f(x)\right)$ and $\mathcal{F}\left(x^n f(x)\right) = (i\partial_k)^n \mathcal{F}\left(f(x)\right)$. Letting $\hat{\rho}\left(\mathbf{k_q}, \mathbf{k_p}\right) = \mathcal{F}_{\mathbf{p}}\mathcal{F}_{\mathbf{q}}\rho(\mathbf{q}, \mathbf{p})$, and expanding the operators into power series and performing Fourier transforms term by term, we obtain

$$\begin{aligned}\mathcal{F}_{\mathbf{p}}\mathcal{F}_{\mathbf{q}}\exp(-\xi\mathbf{p}.\partial_{\mathbf{q}})\rho(\mathbf{q}, \mathbf{p}) &= \hat{\rho}(\mathbf{k_q}, \mathbf{k_p}) + \xi\mathbf{k_q}.\partial_{\mathbf{k_p}}\hat{\rho} + \xi^2\left(\mathbf{k_q}.\partial_{\mathbf{k_p}}\right)^2\hat{\rho} + \ldots \\ &= \exp\left(\xi\mathbf{k_q}.\partial_{\mathbf{k_p}}\right)\hat{\rho}\left(\mathbf{k_q}, \mathbf{k_p}\right) = \hat{\rho}(\mathbf{k_q}, \mathbf{k_p} + \xi\mathbf{k_q}).\end{aligned} \tag{6}$$

Defining the projection operator $\mathcal{P}_{\mathbf{k_p}}$ as the operation that sets $\mathbf{k_p} = 0$, Eq. (5) can be computed by

$$\begin{aligned}\mathcal{F}_{\mathbf{q}}\left(I'(\mathbf{q})\right) &= 2\pi\mathcal{P}_{\mathbf{k_p}}\mathcal{F}_{\mathbf{q}}\mathcal{F}_{\mathbf{p}}\rho(\mathbf{q} - \xi\mathbf{p}, \mathbf{p}) \\ &= 2\pi\mathcal{P}_{\mathbf{k_p}}\hat{\rho}(\mathbf{k_q}, \mathbf{k_p} + \xi\mathbf{k_q}) = 2\pi\hat{\rho}(\mathbf{k_q}, \xi\mathbf{k_q}) \\ \Rightarrow I'(\mathbf{q}) &= 2\pi\mathcal{F}_{\mathbf{q}}^{-1}\hat{\rho}(\mathbf{k_q}, \xi\mathbf{k_q}).\end{aligned} \tag{7}$$

Hence, an image is the inverse 2D Fourier transform, $\mathcal{F}_{\mathbf{q}}^{-1}$, of a slice of $\hat{\rho}(\mathbf{k_q}, \mathbf{k_p})$. It can also be seen that any in-focus image from a particular 4D light field is 3D subspace [5] parameterised by $\mathbf{k_q} = (k_x, k_y)^T$ and ξ.

4. Conclusion

In conclusion, we have demonstrated some mathematical tools to analyze optical systems designed for capturing the geometrical optical phase space density, which can then be used to perform post capture operations, such as refocus, in the spatial or frequency domain.

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