

# New Impulse(Noncausality) Test for Descriptor Systems by Möbius Transformation

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**Abstract:** Descriptor systems (DSs) are usually used to model very-large-scale integration (VLSI) circuit systems and multibody dynamics macromodeling. The analysis of DSs, however, is much more complicated than linear time-invariant (LTI) systems due to the poles at infinity. Möbius transformation (MT) provides a way to transform poles at infinity to finite poles and largely facilitates the reuse or adaptation of the standard techniques for LTI system to analyze DSs. Nonetheless, MT is well known in the literature and its potential use is currently less appreciated in the analysis of DSs. This paper gives a new way to the impulse (noncausality) test using the properties of the transformed LTI systems by MT. Moreover, the applications to the analysis of controllability, observability and regularity are given. Numerical examples are included to show the effectiveness of the proposed method.

**Key Words:** Impulse, Causality, Descriptor system (DS), Möbius transformation (MT), Linear time invariant (LTI) system.

## 1 Introduction

Descriptor systems (DSs), also called singular systems, generalized state space systems, represent dynamic systems well because they contain not only the dynamic part but also nondynamic part. They have been found many applications in very-large-scale integration (VLSI) circuits and multibody dynamics macromodeling [1, 2] and network analysis [3]. A great deal of attention has been devoted to the study of DSs, such as regularization [4], controllability [5], impulse controllability and observability [6], stability and stabilization [7–10], robust pole assignment [11], partial realization [12],  $H_\infty$  control [13–15], positive realness and passivity [1, 16–19],  $H_2$  control [20],  $H_\infty$  filtering [21], and model order reduction (MOR) [22–24].

Impulse (noncausality) analysis of DSs is a very important problem because impulse (noncausality) may cause degradation in performance, damage components, or even destroy the system. Various approaches have been proposed in the literature to address this problem [3, 25–29]. One particular approach relies on the system decompositions by rank decomposition or singular value decomposition (SVD) [3, 25–27]. The technique based on the generalized eigenvalue problem is used in [28]. A new concept, impulsive direction is introduced for the impulse (noncausality) analysis in [29]. However, it is well known that the impulse (noncausality) is equivalent to the existence of poles at infinity, which can be obtained by solving a generalized eigenvalue problem. Until now, no effort has been devoted to use the properties of poles at infinity to study the impulse (noncausality).

In this paper, the main focus is on transforming DSs to linear time-invariant (LTI) systems by Möbius transformation (MT) (also called homographic transformation, linear fractional transformation, or fractional linear transformation). The impulse (noncausality), controllability, observability and regularity are analyzed by the transformed LTI systems. Section 2 reviews the basics of MT and gives the transformed LTI systems. Section 3 details the main results

about the analysis of impulse (noncausality), controllability, observability and regularity in terms of the transformed LTI systems. Numerical examples to demonstrate the effectiveness of the proposed results are given in Section 4. Finally, Section 5 draws the conclusion.

**Notation:** Throughout this paper,  $M^T$  represents the transpose of the matrix  $M$ . Identity matrices are invariably denoted by  $I$  when their dimensions are obvious, otherwise denoted by  $I_n$  to represent an  $n \times n$  identity matrix. In the same way, zero matrices are denoted by  $0$  or  $0_{m \times n}$ . If not explicitly stated, matrices are assumed to have compatible dimensions.  $\mathbb{C}$  represents the complex numbers and  $\mathbb{R}$  denotes the real numbers.

## 2 Preliminaries

Throughout this paper, a DS in the state-space form is assumed:

$$\Sigma : \quad \begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $u(t) \in \mathbb{R}^m$  is the control input and  $y(t) \in \mathbb{R}^l$  is the measured output,  $E, A, B, C$  and  $D$  are appropriately dimensioned real constant matrices with singular  $E$  and  $\text{rank}[E] = q < n$ . If  $\delta(t)$  denotes the differential operator,  $\dot{x}(t)$ , DS  $\Sigma$  represents a continuous-time DS. Or if  $\delta(t)$  denotes the one-step forward operator  $x(t+1)$ , DS  $\Sigma$  is a discrete-time DS. The above system is also identified by DS  $(E, A, B, C, D)$ .

DS  $(E, A, B, C, D)$  is assumed to be *regular*, i.e. there exists a scalar  $s_0 \in \mathbb{C}$  such that  $\det(s_0E - A) \neq 0$ . The finite poles are determined by  $\det(sE - A) = 0$ , at finite  $s$  and pole at infinity is defined as  $\det(sE - A) = 0$ , at infinity, which in turn is isomorphic to  $\det(\frac{1}{s}E - A) = 0$ , at  $s = 0$ . It is said to be impulse-free (causal) or have no infinite poles if  $\deg \det(sE - A) = \lim \text{rank}[E]$  [3, 25]. The existence of poles at infinity for DS is the key difference from the LTI system.

An MT  $\tau(z)$

$$s = \tau(z) = \frac{az + b}{cz + d}, \quad s \in \mathbb{C}, \quad (1)$$

This work was supported by Small Project Funding of HKU from HKU SPACE Research Fund under grant 201007176165.

will be used in the following lemma, where  $a, b, c, d \in \mathbb{R}$  satisfying  $ad - bc \neq 0$ .

**Assumption 1**  $a$  and  $c$  are chosen such that  $aE - cA$  is nonsingular.

**Lemma 1** DS  $(E, A, B, C, D)$  is transformed to an LTI system  $(I, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  by the MT  $\tau(z)$  in (1), i.e.

$$\tau(E, A, B, C, D) = (I, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}),$$

where

$$\tilde{A} = (aE - cA)^{-1} (dA - bE), \quad (2)$$

$$\tilde{B} = (aE - cA)^{-1} B, \quad (3)$$

$$\tilde{C} = (ad - cb) C (aE - cA)^{-1} E, \quad (4)$$

$$\tilde{D} = cC (aE - cA)^{-1} B + D. \quad (5)$$

**Proof.** By replacing  $s$  in (1), the transfer function of the DS  $(E, A, B, C, D)$  can be rewritten as

$$\begin{aligned} & C(sE - A)^{-1} B + D \\ &= C \left( \left( \frac{az + b}{cz + d} \right) E - A \right)^{-1} B + D \\ &= C(cz + d) (z(aE - cA) - dA + bE)^{-1} B + D \\ &= \tilde{D} + \tilde{C} (z - \tilde{A})^{-1} \tilde{B}, \end{aligned}$$

due to the invertible assumption of  $aE - cA$ . The result follows. ■

When the MT  $\tau(z)$  in (1) reduces to the typical MT

$$\tau_{\text{typical}}(s) = \frac{z - 1}{z + 1}, \quad (6)$$

the following corollary can be get from Lemma 1 immediately.

**Corollary 1** By the typical MT  $\tau_{\text{typical}}(s)$  in (6), DS  $(E, A, B, C, D)$  becomes LTI system  $(I, \bar{A}, \bar{B}, \bar{C}, \bar{D})$ , i.e.,

$$\tau_{\text{typical}}(E, A, B, C, D) = (I, \bar{A}, \bar{B}, \bar{C}, \bar{D}),$$

where

$$\bar{A} = (E - A)^{-1} (A + E), \quad (7)$$

$$\bar{B} = (E - A)^{-1} B, \quad (8)$$

$$\bar{C} = 2C(E - A)^{-1} E, \quad (9)$$

$$\bar{D} = C(E - A)^{-1} B + D. \quad (10)$$

**Remark 1** In the case of  $E = I$ , Corollary 1 reduces to the standard result for transforming a continuous-time LTI system  $(I, A, B, C, D)$  to a discrete-time LTI system  $(I, \hat{A}, \sqrt{2}(I - A)^{-1} B, \sqrt{2}C(I - A)^{-1}, \hat{D})$ , i.e.

$$\tau_{\text{typical}}(I, A, B, C, D) = (I, \hat{A}, \hat{B}, \hat{C}, \hat{D}),$$

where

$$\hat{A} = (I - A)^{-1} (A + I),$$

$$\hat{B} = \sqrt{2}(I - A)^{-1} B,$$

$$\hat{C} = \sqrt{2}C(I - A)^{-1},$$

$$\hat{D} = C(I - A)^{-1} B + D.$$

**Remark 2** It is very clear that infinity is transformed to a finite point by the MT in (1), which avoids the difficulties for handling infinity and facilitates the use of the properties of infinity. It is well known that the existence of poles at infinity, also the source of impulse (noncausality), distinguishes DSs from LTI systems. So the impulse (noncausality) can be tested by the transformed LTI system in terms of the properties of poles at infinity. Moreover, when the typical MT in (6) is considered, the open left half plane,  $S_0 = \{s | \text{Re}(s) < 0\}$  is restricted to the open unit disk  $|z| < 1$ , which is much smaller than  $S_0$ . Direct applications include the stability analysis and the frequency domain analysis with very wide range frequencies for continuous time DS.

### 3 Impulse (Noncausality) Analysis

#### 3.1 Poles

Under the regularity assumption of the DS  $(E, A, B, C, D)$ , there exist two real nonsingular matrices  $P$  and  $Q$  such that

$$PEQ = \begin{bmatrix} I_p & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} J & 0 \\ 0 & I_{n-p} \end{bmatrix}, \quad (11)$$

where  $N$  is a nilpotent matrix with nilpotent index  $h$ , (i.e.  $N^{h-1} \neq 0, N^h = 0$ ).  $h$  is also called the index of the matrix pencil  $\{E, A\}$  or the index of the DS  $(E, A, B, C, D)$ . Then the finite poles  $s_i, i = 1, 2, \dots, p$ , are the eigenvalues of matrix  $J$  and DS  $(E, A, B, C, D)$  is impulse-free (causal) if and only if  $N = 0$  [3]. It is pretty easy to show that finite poles  $s_i, i = 1, 2, \dots, p$ , becomes

$$z_i = \tau^{-1}(s_i) = \frac{s_i d - b}{-s_i c + a}, \quad i = 1, 2, \dots, p, \quad (12)$$

by the MT  $\tau(z)$  in (1).

From Lemma 1, we have the following results. The eigenvalues of  $\tilde{A}$  are also called the poles of LTI system  $(I, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ .

**Theorem 1** The following statements hold.

- 1) The transformed poles  $z_i, i = 1, 2, \dots, p$ , in (12) are the eigenvalues of  $\tilde{A}$  in (2).
- 2) The transformed pole  $\alpha = -\frac{d}{c}$  from infinity is the repeated eigenvalue of  $\tilde{A}$  with  $n - p$  multiplicities.

**Proof.** (1) As the transformed poles  $z_i \neq -\frac{d}{c}, i = 1, 2, \dots, p$ , are finite, we have

$$\det \left( \frac{1}{cz_i + d} I_n \right) \neq 0, \quad i = 1, 2, \dots, p. \quad (13)$$

From (12) and (13), and together with  $\det(s_i E - A) = 0$ , it follows that

$$\begin{aligned} & \det(s_i E - A) \\ &= \det(aE - cA) \det \left( \frac{1}{cz_i + d} I_n \right) \det(z_i I - \tilde{A}) \\ &= 0, \end{aligned} \quad (14)$$

which further implies

$$\det(z_i I - \tilde{A}) = 0,$$

due to  $\det(aE - cA) \neq 0$  in Assumption 1. Thus,  $z_i, i = 1, 2, \dots, p$ , are eigenvalues of  $\tilde{A}$ .

(2) It is pretty easy to show that

$$\det(\alpha I - \tilde{A}) = \det((bc - ad)E) = 0, \quad (15)$$

due to the singularity of  $E$  and  $bc - ad \neq 0$ . So,  $\alpha$  is an eigenvalue of  $\tilde{A}$ . Furthermore, the multiplicity of the eigenvalue  $\alpha$  is  $n - p$ . If not, assume that  $\tilde{A}$  has another finite eigenvalue  $z'_0$  except  $z_i, i = 1, 2, \dots, p$  in (12) and  $\alpha$ . Then  $s'_0 = \tau(z'_0, a, b, c, d)$ , is a finite pole of DS  $(E, A, B, C, D)$  from (14), which gives that DS  $(E, A, B, C, D)$  has  $p + 1$  finite poles. This is obviously wrong. Therefore,  $\tilde{A}$  has eigenvalue  $\alpha$  with  $n - p$  multiplicities. ■

**Remark 3** From Theorem 1, we know that whether DS is impulse-free (causal) or not,  $\tilde{A}$  always has repeated eigenvalues  $\alpha$  with  $n - p$  multiplicities. However, the properties of eigenvectors associated with eigenvalue  $\alpha$  can distinguish between impulse-free (causal) DS and impulse (noncausal) DS, shown in Theorem 2.

So, together with Theorem 1 and Corollary 1, we have the following corollary for the stability test of continuous-time DS as open left half plane is changed to the open unit disk by the typical MT in (6). Due to the compact unit disk, the analysis of the poles is very handy under the case of wide range of the poles of continuous-time DS. Better estimation of the range of the poles helps to speed up the simulation when large scale DS is considered.

**Corollary 2** Continuous-time DS  $(E, A, B, C, D)$  is stable if and only if  $\tilde{A}$  in (7) has  $n - p$  repeated eigenvalues  $-1$  and the others of  $\tilde{A}$  belong to the open disk  $|z| < 1$ .

### 3.2 Impulse (Noncausality)

**Theorem 2** We have the following conclusions.

- 1) DS  $(E, A, B, C, D)$  has impulse (is noncausal) if and only if the eigenvectors,  $p_i, i = 1, 2, \dots, n - p$ , associated with the repeated eigenvalues  $\alpha$  ( $\tilde{A}p_i = \alpha p_i$ ) are linearly dependent.
- 2) DS  $(E, A, B, C, D)$  is impulse-free (causal) if and only if  $p_i, i = 1, 2, \dots, n - p$ , are linearly independent.

**Proof.** (1)

$$\tilde{A}p_i = \alpha p_i, \quad i = 1, \dots, n - p,$$

gives

$$\alpha(aE - cA)p_i = (dA - bE)p_i,$$

which results in

$$\frac{bc - ad}{c}Ep_i = 0.$$

Due to  $ad - bc \neq 0$ , it follows that  $Ep_i = 0$ . Thus,  $p_i, i = 1, 2, \dots, n - p$ , are also the eigenvectors of  $E$  associated with eigenvalue 0. From (11), we have

$$P^{-1} \begin{bmatrix} I_p & 0 \\ 0 & N \end{bmatrix} Q^{-1}p_i = 0,$$

which further gives

$$p_i = Q \begin{bmatrix} 0 \\ p_{N_i} \end{bmatrix}, \quad Np_{N_i} = 0, \quad i = 1, \dots, n - p, \quad (16)$$

with  $NV_N = 0$ , where

$$V_N = \left[ \begin{bmatrix} 0 \\ p_{N_1} \end{bmatrix} \quad \cdots \quad \begin{bmatrix} 0 \\ p_{N_{n-p}} \end{bmatrix} \right].$$

It follows that  $V_N$  belongs the null space of  $N$  and

$$\text{rank}(V_N) = n - p - \text{rank}(N) \leq n - p$$

due to  $N \neq 0$ . Therefore,  $p_{N_i}, i = 1, 2, \dots, n - p$ , are linearly dependent.

Conversely, the linear dependence of  $p_i, i = 1, 2, \dots, n - p$ , implies the linear dependence of  $p_{N_i}, i = 1, 2, \dots, n - p$ . So,  $N$  is not invertible and  $N \neq 0$ . So the DS  $(E, A, B, C, D)$  has impulse.

(2) The proof can be implied from the proof of (1). ■

**Remark 4** The connection between the properties of finite poles of the transformed LTI system and the test of impulse (noncausality) is first established, which overcomes the difficulties for dealing with poles at infinity. This new insight provides a fresh tool on the test of impulse (noncausality).

From the results in [6], the impulse (noncausality) analysis is enriched by the results in Theorem 2, which shown in the following corollary. Methods 2 and 3 are not practical as all  $s \in \mathbb{C}$  are needed to be checked. Method 4 also fails when  $A$  is close to be singular, which can be avoided by free choices of  $a, b, c$  and  $d$  in (1) by the proposed method. Methods 5 and 6 also fails in the some cases, which are shown Example 2 in Section 5.

**Corollary 3** The following statements are equivalent:

- 1) DS  $(E, A, B, C, D)$  is impulse-free;
- 2) If there exist a vector  $v \in \mathbb{R}^n$  and a vector  $w \in \mathbb{R}^n$  such that  $(sE - A)v = Ew$ , for all  $s \in \mathbb{C}$  then  $v = 0$ ;
- 3) If there exist a vector  $v \in \mathbb{R}^n$  and a vector  $w \in \mathbb{R}^n$  such that  $Ev = 0$  and  $Av = Ew$ , for all  $s \in \mathbb{C}$  then  $v = 0$ ;
- 4)  $(A^{-1}ImE) \cap KerE = \{0\}$ .
- 5) The block  $A_{22}$  either is nonsingular or vanishes, where  $WET = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}$ ,  $WAT = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ , where  $W$  and  $T$  are invertible matrices.
- 6)  $\text{rank} \begin{bmatrix} 0 & E \\ E & A \end{bmatrix} = n + \text{rank}E$ ;
- 7)  $p_i, i = 1, 2, \dots, n - p$ , associated with the repeated eigenvalues  $\alpha$  for matrix  $\tilde{A}$  are linearly independent.

The index of matrix pencil  $\{E, A\}$  is also implied from the properties of vectors  $p_i, i = 1, 2, \dots, n - p$ , which can be further applied to the passivity test for DS [1, 19, 30].

**Corollary 4** The index of DS  $(E, A, B, C, D)$ ,  $h$ , is equivalent to the maximum number of proportional vectors of  $p_i, i = 1, 2, \dots, n - p$ .

**Proof.** For matrix  $\tilde{A}$ , there exists a matrix  $V_\alpha$  such that  $\tilde{A}V_\alpha = V_\alpha D_\alpha$ , where

$$D_\alpha = \text{diag}(\alpha, \dots, \alpha),$$

and the columns of  $V_\alpha$  are the eigenvectors associated with the repeated eigenvalue  $\alpha$ . From (16),  $V_\alpha$  can be expressed

as

$$\begin{aligned} V_\alpha &= [p_1 \ \cdots \ p_{n-p}] \\ &= Q \left[ \begin{bmatrix} 0 \\ p_{N_1} \end{bmatrix} \ \cdots \ \begin{bmatrix} 0 \\ p_{N_{n-p}} \end{bmatrix} \right]. \end{aligned}$$

Now we need to prove that some of  $p_{N_i}$  are proportional. Firstly for matrix  $N$ , there exists a similarity transformation  $T$  satisfying

$$TNT^{-1} = \begin{bmatrix} N_1 & & 0 \\ & \ddots & \\ 0 & & N_r \end{bmatrix},$$

where

$$N_i = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \in R^{r_{N_i} \times r_{N_i}},$$

has ones on the superdiagonal and is called a nilpotent Jordan block with  $\sum_{i=1}^r r_{N_i} = n - p$ . It is clear that the nilpotent index for  $N_i$  is  $r_{N_i}$ ,  $i = 1, 2, \dots, r$ , and the eigenvectors  $p_{N_i}$ ,  $i = 1, 2, \dots, n - p$ , associated with the repeated eigenvalue 0 of matrix  $N_i$  are

$$p_{N_i} = k [1 \ 0 \ \cdots \ 0]^T \in R^{r_{N_i} \times 1}, \ k \neq 0,$$

which follows that the columns of

$$V_{N_i} = [p_{N_i} \ \cdots \ p_{N_i}] \in R^{r_{N_i} \times r_{N_i}}$$

are proportional with  $N_i V_{N_i} = 0$ . Moreover, we get

$$V_\alpha = Q \begin{bmatrix} I & 0 \\ 0 & T^{-1} \end{bmatrix} [\tilde{V}_{N_1} \ \cdots \ \tilde{V}_{N_r}],$$

where

$$\tilde{V}_{N_i} = \left[ \begin{bmatrix} 0 \\ p_{N_i} \end{bmatrix} \ \cdots \ \begin{bmatrix} 0 \\ p_{N_i} \end{bmatrix} \right], \quad i = 1, \dots, r.$$

which shows that the columns of  $V_\alpha$  are proportional. So, we have

$$h = \max_i r_{N_i} = \max_i \{\text{row}(\tilde{V}_{N_i})\}.$$

i.e. the maximum number of columns, which are proportional. ■

## 4 Application to Regularity, Controllability and Observability

### 4.1 Regularity

**Corollary 5** *DS  $(E, A, B, C, D)$  is regular if and only if the LTI system  $(I, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is regular.*

**Proof.** Due to  $\det(aE - cA) \neq 0$ , if there exists a finite scalar  $s_0 \neq \frac{a}{c}$  such that  $\det(s_0E - A) \neq 0$ , then

$$z_0 = \tau^{-1}(s_0, a, b, c, d) = \frac{s_0 d - b}{-s_0 c + a},$$

is finite and

$$\det\left(\frac{1}{cz_0 + d} I_n\right) \neq 0.$$

From (14), it follows that

$$\det(z_0 I - \tilde{A}) = \frac{\det(s_0 E - A)}{\det(aE - cA) \det\left(\frac{1}{cz_0 + d} I_n\right)} \neq 0.$$

So, LTI system  $(I, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is regular.

Conversely, if there exists a finite scalar  $z_0 \neq \alpha$ , such that  $\det(z_0 I - \tilde{A}) \neq 0$ , then we have

$$\det(s_0 E - A) = \det(\tau(z_0) E - A) \neq 0.$$

The conclusion holds. ■

### 4.2 Controllability

**Corollary 6** *DS  $(E, A, B, C, D)$  is controllable if and only if the LTI system  $(I, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is controllable.*

**Proof.** The LTI system  $(I, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is controllable if and only if for any finite  $z \in \mathbb{C}$ ,

$$\begin{aligned} &\text{rank} [zI_n - \tilde{A} \ \tilde{B}] \\ &= \text{rank} [(az + b)E - (cz + d)A \ B] = n, \end{aligned}$$

So if  $cz + d \neq 0$ , we have

$$\begin{aligned} &\text{rank} [zI_n - \tilde{A} \ \tilde{B}] \\ &= \text{rank} \left[ \begin{pmatrix} az+b \\ cz+d \end{pmatrix} E - A \ B \right] = n, \end{aligned}$$

which results in

$$\text{rank} [sE - A \ B] = n, \quad \forall s \in \mathbb{C}, \ s \text{ finite.} \quad (17)$$

If  $cz + d = 0$ , it is easy to obtain that  $z = \alpha$ , and  $\text{rank} [(a\alpha + b)E \ B] = \text{rank} [E \ B] = n$ , together with (17) gives the controllability of DS  $(E, A, B, C, D)$  [3]. ■

### 4.3 Observability

**Corollary 7** *The observability of DS  $(E, A, B, C, D)$  is equivalent to the controllability of LTI system  $(I, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ , where*

$$\begin{aligned} \mathcal{A} &= (aE^T - cA^T)^{-1} (dA^T - bE^T), \\ \mathcal{B} &= (aE^T - cA^T)^{-1} C^T, \\ \mathcal{C} &= (ad - bc) B^T (aE^T - cA^T)^{-1} E^T, \\ \mathcal{D} &= cB^T (aE^T - cA^T)^{-1} C^T + D^T. \end{aligned}$$

**Proof.** DS  $(E, A, B, C, D)$  is observable if and only if DS  $(E^T, A^T, C^T, B^T, D^T)$  is controllable. From Corollary 6, DS  $(E^T, A^T, C^T, B^T, D^T)$  is controllable if and only if the LTI system  $(I, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$  is controllable. This completes the proof. ■

## 5 Numerical Examples

**Example 1:** A very simple example is given to show the ways to check the properties of DS by the transformed

LTI system. The system matrices of a continuous-time DS  $(E, A, B, C, D)$  are given by

$$E = \begin{bmatrix} 3.1326 & 0.1529 & 2.6503 & -1.9745 \\ -1.3168 & 0.9076 & -1.6281 & 1.3934 \\ -2.2474 & -0.1597 & -1.7871 & 2.2033 \\ 0.1058 & -0.0061 & 0.0761 & -0.2531 \end{bmatrix},$$

$$A = \begin{bmatrix} -12.3211 & -0.5476 & -12.3852 & -1.3960 \\ 3.8224 & -3.8348 & 6.4578 & -0.0443 \\ 10.1039 & 0.7359 & 10.2075 & 1.0874 \\ -0.5296 & 0.0220 & -0.5178 & 0.9484 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}^T, D = 0,$$

$$C = \begin{bmatrix} 4.0132 & 0.5475 & 4.2020 & 0.6321 \\ 0.4729 & 0.1873 & 0.5194 & 1.7128 \end{bmatrix}.$$

For  $E$  and  $A$ , there exist

$$P = \begin{bmatrix} 2.0010 & 0.2483 & 1.7638 & 0.2245 \\ 0.2483 & 0.1489 & 0.1502 & 0.0384 \\ 1.7638 & 0.1502 & 2.2880 & 0.3692 \\ 0.2245 & 0.0384 & 0.3692 & 1.6744 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1.9547 & -1.8721 & -1.3984 & 0.0892 \\ -1.8721 & 8.9893 & 0.8771 & -0.1485 \\ -1.3984 & 0.8771 & 1.4833 & -0.1597 \\ 0.0892 & -0.1485 & -0.1597 & 0.6239 \end{bmatrix},$$

such that

$$PEQ = \begin{bmatrix} I_2 & 0 \\ 0 & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{bmatrix},$$

$$PAQ = \text{diag}(-3, -4, 1, 1).$$

It is clear that DS  $(E, A, B, C, D)$  is stable and has impulse. The index of DS  $(E, A, B, C, D)$  is 2. Furthermore, due to

$$\text{rank} \begin{bmatrix} A & AB & A^2B & A^3B \end{bmatrix} = 4,$$

$$\text{rank} \begin{bmatrix} E & B \end{bmatrix} = 4,$$

and

$$\text{rank} \begin{bmatrix} C^T & (CA)^T & (CA^2)^T & (CA^3)^T \end{bmatrix}^T = 4,$$

$$\text{rank} \begin{bmatrix} E^T & C^T \end{bmatrix}^T = 4,$$

continuous-time DS  $(E, A, B, C, D)$  is controllable and observable.

Using the typical MT in (6), we can get the system matrices of the transformed LTI system  $(\bar{I}, \bar{A}, \bar{B}, \bar{C}, \bar{D})$

$$\bar{A} = \begin{bmatrix} 1.3977 & 0.2386 & 2.6441 & 4.8736 \\ -1.3739 & -0.7642 & -1.7587 & -3.0093 \\ -1.9780 & -0.2353 & -3.2759 & -5.1107 \\ 0.1462 & 0.0145 & 0.1877 & -0.4575 \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 10.9377 & 7.6183 \\ -6.6118 & -4.4193 \\ -10.8779 & -7.6427 \\ -0.2708 & -0.4783 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 0.6508 & 0.1070 & 0.2035 & -3.2211 \\ 0.0993 & 0.0596 & 0.0601 & 0.0154 \end{bmatrix},$$

$$\bar{D} = \begin{bmatrix} -5.6044 & -4.2627 \\ -2.1807 & -2.0145 \end{bmatrix}.$$

• The eigenvalues of  $\bar{A}$  are  $-1, -1, -0.5$  and  $-0.6$ . From Corollary 2, continuous-time DS  $(E, A, B, C, D)$  is stable.

• The eigenvectors associated with pole  $\alpha = -1$  are

$$p_1 = \begin{bmatrix} 0.6285 & -0.3942 & -0.6667 & 0.0718 \end{bmatrix}^T,$$

$$p_2 = \begin{bmatrix} -0.6285 & 0.3942 & 0.6667 & -0.0718 \end{bmatrix}^T,$$

which are linearly dependent. This shows that DS  $(E, A, B, C, D)$  has impulse from Theorem 2. Moreover, the index of DS  $(E, A, B, C, D)$  is 2 as two eigenvectors  $p_1$  and  $p_2$  are linearly dependent.

• It is very easy to obtain that

$$\text{rank} \begin{bmatrix} \bar{A} & \bar{A}\bar{B} & \bar{A}^2\bar{B} & \bar{A}^3\bar{B} \end{bmatrix} = 4,$$

which results in that DS  $(E, A, B, C, D)$  is controllable from Corollary 6.

• The LTI system  $(I, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$  from Corollary 7 is observable from

$$\text{rank} \begin{bmatrix} \mathcal{C}^T & (\mathcal{C}\mathcal{A})^T & (\mathcal{C}\mathcal{A}^2)^T & (\mathcal{C}\mathcal{A}^3)^T \end{bmatrix}^T = 4,$$

where

$$\mathcal{A} = \begin{bmatrix} 1.3977 & -1.3739 & -1.9780 & 0.1462 \\ 0.2386 & -0.7642 & -0.2353 & 0.0145 \\ 2.6441 & -1.7587 & -3.2759 & 0.1877 \\ 4.8736 & -3.0093 & -5.1107 & -0.4575 \end{bmatrix},$$

$$\mathcal{B} = \begin{bmatrix} -1.4384 & -0.1748 \\ -0.0967 & -0.0086 \\ -2.1862 & -0.3392 \\ -1.9798 & -1.6667 \end{bmatrix},$$

$$\mathcal{C} = \begin{bmatrix} 9.9153 & -6.1419 & -9.3647 & 0.8764 \\ 7.7562 & -4.5322 & -7.6219 & 0.7447 \end{bmatrix},$$

$$\mathcal{D} = \begin{bmatrix} -5.6044 & -2.1807 \\ -4.2627 & -2.0145 \end{bmatrix},$$

So we can get that DS  $(E, A, B, C, D)$  is observable.

**Example 2:** This example will show that methods 5 and 6 in Corollary 3 fail to analyze the impulse (noncausality) when  $E$  and  $A$  are

$$E = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 400I_3 & 0 \\ 0 & 10^{-8}\text{diag}(0.1, 1, 0.4) \end{bmatrix}.$$

It is obvious that it is impulse-free. However,

$$A_{22} = 10^{-8}\text{diag}(0.1, 1, 0.4),$$

is close to be singular due to  $\det(A_{22}) = 4 \times 10^{-26}$ , which gives the impulse conclusion from method 5 in Corollary 3. And

$$\text{rank} \begin{bmatrix} 0 & E \\ E & A \end{bmatrix} = 6 < n + \text{rank}E = 9,$$

also gives the same conclusion from method 6 in Corollary 3. However, by the typical MT, we have

$$\bar{A} = \begin{bmatrix} -1.005I_3 & 0 \\ 0 & -I_3 \end{bmatrix}.$$

It is easy to show that the eigenvectors associated with  $-1$  are linearly independent, which gives the impulse-free conclusion.

## 6 Conclusions

The MT is used to transform DSs to LTI systems, which permits us to use the established techniques for the LTI systems to study DSs. The connection between the properties of DSs and the properties of the transformed LTI systems is first found. The poles at infinity are changed to finite poles in the transformed LTI systems for better using their properties. This facilitates the analysis of DSs as the existence of poles at infinity makes them different from LTI system. The dependence of eigenvectors associated with the finite poles transforming from the poles at infinity provides a novel solution for the test of the impulse (noncausality). The controllability, observability and regularity also can be checked by the corresponding counterparts of the transformed LTI system. Further attention will focus on the impulse control, passivity test and stabilization in terms of the transformed LTI system, and engineering application with more realistic case studies as impulse (noncausality) is not allowed for practical system.

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