Statistical modelling of the wavelet coefficients with different bases and decomposition levels

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Abstract: Various wavelet coefficient statistics are useful to increase the compression efficiency of images. The distribution of the wavelet coefficients within a sub-band affects how the decompression values should be adjusted. A generalised Gaussian distribution model is seen to be applicable from a theoretical point of view, and a convenient way is provided to estimate the parameters from the empirical data. This knowledge is applied to examine the best statistical modelling of the wavelet coefficients for different wavelet bases and decomposition levels.

1 Introduction

Multimedia transmissions have become increasingly popular in a diverse array of applications. In order to obtain a good compression of video and image signals, many international standards are being developed, including the JPEG 2000 standard for image compression, with higher efficiency and more advanced features [1]. Wavelet transforms are at the core of this method. Compression is primarily achieved by concentrating the energy of the image into a few wavelet coefficients, while their statistics are also taken into account for further compression gain.

Two kinds of wavelet statistics are found useful [2]. The interscale dependencies have been effectively employed in a variety of tree-structured coding techniques, such as SPIHT [3]. Within the same sub-band, the statistical distribution of the wavelet coefficients has also received much notice. In the absence of any prior statistical distribution information about the coefficients, the decoded value is set to be the mid-point of the codeblock. However, the best decoded value should be the centroid [4]. For any leptokurtic or fat-tail distributions, it is known that the centroid is smaller than the mid-point [5], yet a more precise modelling of the coefficients is needed to determine the optimal shift from the mid-point in decoding. For actual experimental data, generalised Gaussian distribution is a powerful model for the coefficient distributions because it includes a range of kurtosis values by varying a parameter that controls the shape of the distribution. In this paper, we investigate how it varies with the different number of decomposition levels and wavelet bases for natural images. This is supported by a theoretical analysis on the plausibility of modelling by generalised Gaussian distribution and an empirical study of the values of the shape parameter.

2 Modelling methodology

A doubly stochastic model has been shown to be an effective image model in investigating the transform coefficient distributions [2, 5]. Let the wavelet coefficient be I, a stochastic quantity drawn from a distribution parameterised by ϕ. The variable ϕ can be thought of as the local variance of the wavelet coefficients. The distribution of the wavelet coefficient conditioned on ϕ is considered a zero-mean Gaussian distribution, i.e. $\mathcal{P}(I|\phi) \sim \mathcal{N}(0, \sqrt{\phi})$. ϕ itself is a stochastic process with non-negative values. If ϕ has an exponential distribution, i.e. $\mathcal{P}(\phi) = \lambda \exp\{-\lambda\phi\}$

$$\begin{aligned} \mathcal{P}(I) &= \int_0^\infty \mathcal{P}(I|\phi)\mathcal{P}(\phi)d\phi \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi\phi}} \exp\left\{-\frac{I^2}{2\phi}\right\} \lambda \exp\{-\lambda\phi\} d(\phi) \\ &= \frac{\sqrt{2\lambda}}{2} \exp\left\{-\sqrt{2\lambda}|I|\right\} \end{aligned} \tag{1}$$

In this case, the resulting distribution of I is exactly Laplacian with parameter $\sqrt{2\lambda}$. Indeed, Laplacian distribution has often been used in the coding of wavelet coefficients, among the many fat-tail distributions [6].

Yet, when ϕ is not exponentially distributed, no simple closed-form solution can be found. However, it is known that as long as the distribution of ϕ is not singular, $\mathcal{P}(I)$ must have a larger kurtosis value than a Gaussian, i.e. it is fat-tail. In fact, the larger the variance of the distribution of ϕ, the greater the kurtosis value [5, 7]. To allow for multiple kurtosis values, it is better to use a generalised Gaussian distribution to model wavelet sub-band coefficients [8]. The generalised Gaussian distribution, with zero mean, has the probability density function

$$\mathcal{P}(I) = \frac{v}{2\beta\Gamma\left(\frac{1}{v}\right)} \exp\left\{-\left(\frac{|I|}{\beta}\right)^v\right\} \tag{2}$$

where $v > 0$ controls the shape of the distribution and β the spread. As $v \to \infty$, it approaches the uniform distribution. When $v = 2$ and $\beta = \sqrt{2}\sigma$, it becomes a standard Gaussian distribution. The Laplacian distribution is another special case of the generalised Gaussian distribution, obtained by setting $v = 1$ and $\beta = 1/\lambda$. Empirically, v and β can be determined by computing $\chi = E[|I|]$ and $\psi = E[I^2]$. Because of the relationship

$$\frac{\psi}{\chi^2} = \frac{\Gamma\left(\frac{1}{v}\right)\Gamma\left(\frac{3}{v}\right)}{\Gamma^2\left(\frac{2}{v}\right)} \tag{3}$$

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we can use a look-up table with different values of v and determine its value from (χ, ψ) [9]. After we obtain the value of v, we can get β by $\beta = \psi \Gamma(1/v)/\Gamma(3/v)$.

3 Fitting with empirical data

3.1 Different decomposition levels

Equipped with the technique for finding the shape parameters above, we now analyse the wavelet coefficients from a database of more than 20 natural images at different decomposition levels with different wavelet bases. We focus our attention on natural images only as other image types, such as texts, would exhibit different transform coefficient statistics [10]. Each image is of size 256×256 or 512×512, decomposed to five levels. We test with both the integer wavelet basis and the floating-point wavelet basis specified in part I of the JPEG 2000 standard. For the integer wavelet basis, the low-pass and high-pass filters are, respectively

low-pass filter:

$$s_0[n] = \frac{1}{8}[-1 \quad 2 \quad 6 \quad 2 \quad -1] \tag{4}$$

high-pass filter:

$$s_1[n] = \frac{1}{2}[-1 \quad 2 \quad -1]. \tag{5}$$

Their graphical representations in both time and frequency domains are shown in Fig. 1. For the floating-point wavelet basis, the low-pass and high-pass filters are, respectively

$$\begin{aligned} s_0[n] = [&0.0267487 \quad -0.0168641 \quad -0.0782232 \\ &0.2668641 \quad 0.6029490 \quad 0.2668641 \\ &-0.0782232 \quad -0.0168641 \quad 0.0267487] \end{aligned} \tag{6}$$

$$\begin{aligned} s_1[n] = [&0.0912717 \quad -0.0575435 \quad -0.5912717 \quad 1.1150870 \\ &-0.5912717 \quad -0.0575435 \quad 0.0912717] \end{aligned} \tag{7}$$

Their graphical representations in both time and frequency domains are shown in Fig. 2.

Figure 3 shows the best shape parameters for the LH bands (low-pass in the horizontal direction and high-pass in the vertical direction), HL bands (high-pass in the horizontal direction and low-pass in the vertical direction) and HH bands (high-pass in both directions), at different decomposition levels, using the integer wavelet basis shown above. At the fifth decomposition level, the number of data points is too small (each sub-band is only 8×8 or 16×16) to warrant an accurate fitting of the data. For the first four levels, it is seen that the HH bands have relatively stable shape parameters across different decomposition levels, around 0.7–0.8. Figure 4 shows the same study for the floating-point wavelet basis described above. Although the specific data points are different for the two wavelet bases, the general trend agrees. Overall, in the HH band the shape parameter stays sufficiently close across different sub-bands, so we can use the same bias in wavelet coefficient decoding. However, the LH and HL bands show a steady increase in the shape parameters with more decomposition levels. As a larger kurtosis factor implies better resemblance to a uniform distribution, the bias in reconstruction should therefore reduce accordingly as we increase the decomposition levels.

3.2 Different wavelet bases

While part I of the JPEG 2000 standard specifies only two particular wavelet bases for the transform, in part II one can use custom-designed wavelet bases. Therefore, in this Section, we study how different wavelet bases may affect the shape parameter in the sub-bands. In particular, we investigate how the parameter varies within the same family of wavelets. We study the Daubechies wavelet bases here [11]. They are obtained by spectral factorisation of the same equation with different choice of the parameters and therefore yield similar properties, such as orthogonality, minimum phase and compact support, among the bases in the family.

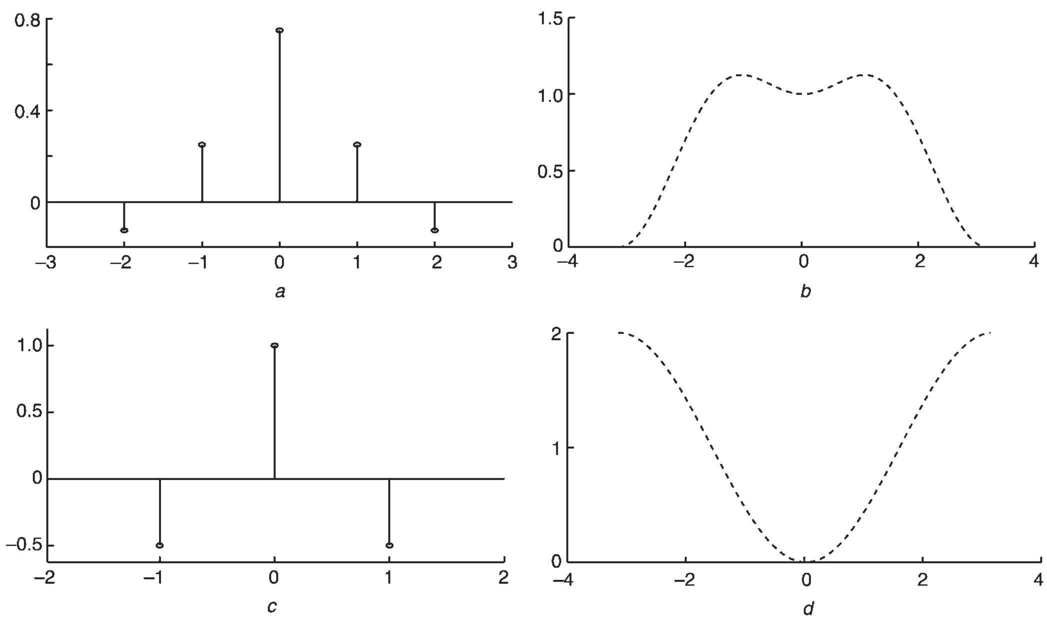

Fig. 1 *Integer wavelet basis in JPEG 2000*

a Time representation of $s_0[n]$
b Frequency representation of $s_0[n]$
c Time representation of $s_1[n]$
d Frequency representation of $s_1[n]$

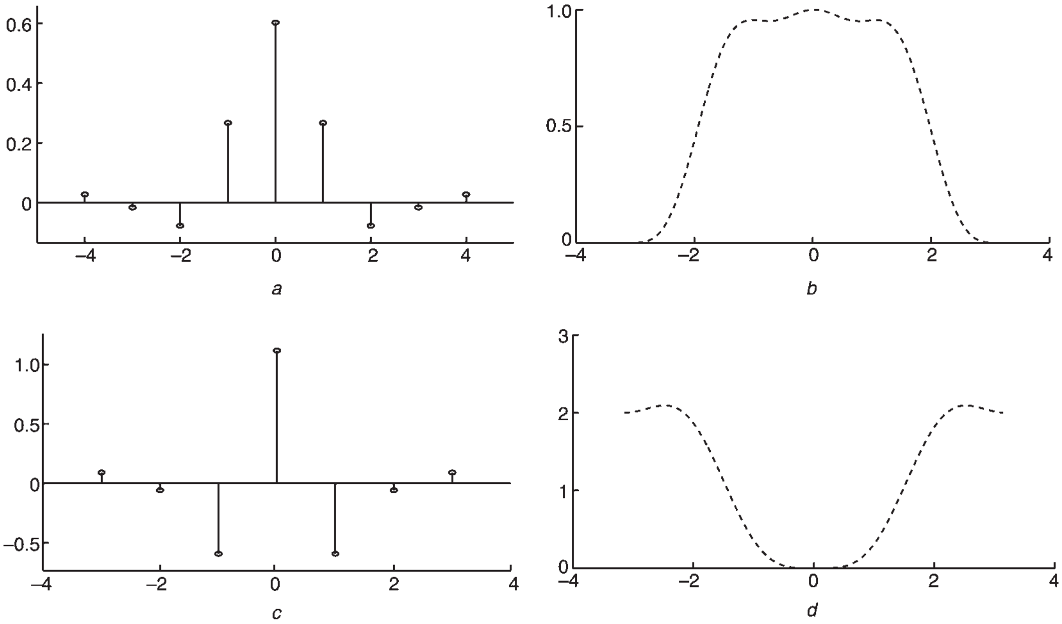

Fig. 2 *Floating-point wavelet basis in JPEG 2000*

a Time representation of $s_0[n]$
b Frequency representation of $s_0[n]$
c Time representation of $s_1[n]$
d Frequency representation of $s_1[n]$

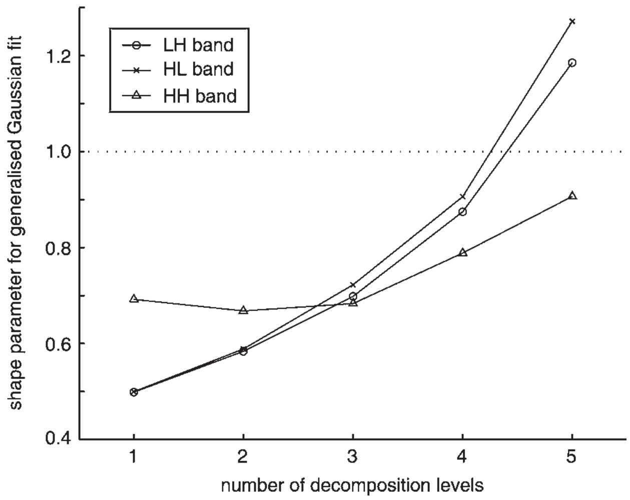

Fig. 3 *Shape parameters of wavelet coefficients using the integer wavelet basis in JPEG 2000*

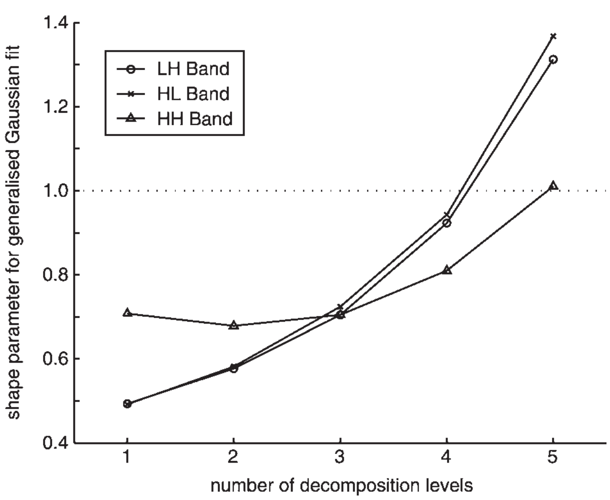

Fig. 4 *Shape parameters of wavelet coefficients using the floating-point wavelet basis in JPEG 2000*

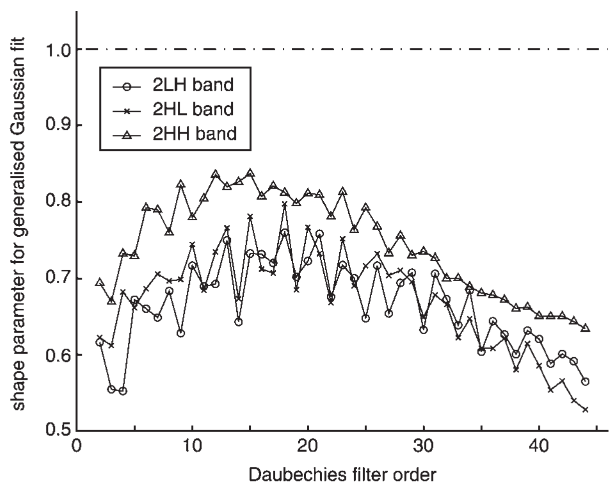

Fig. 5 *Mean value of shape parameters across different wavelet filter orders*

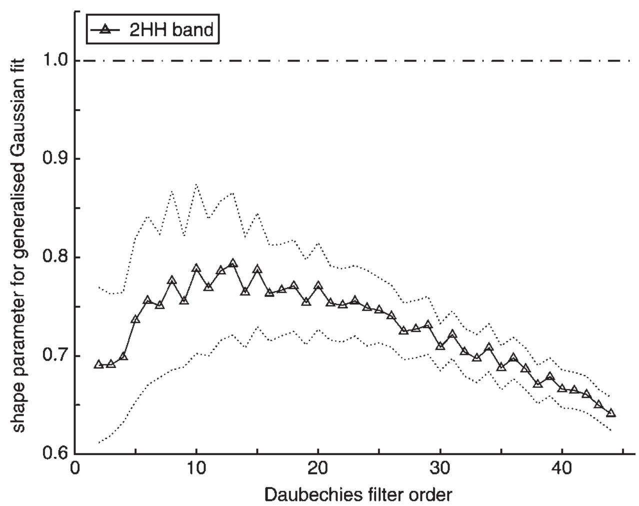

Fig. 6 *Range of shape parameters across different wavelet filter orders*

Figure 5 shows the variation of the shape parameters of a sub-band across different bases. It is interesting to note that there is a zig-zag pattern in the curves, and the same is true when we test different sub-bands and different images. However, they all follow the same general trend of increasing the shape parameters from small to medium filter orders, and then decrease steadily from medium to large filter orders. Figure 6 shows the shape parameters with their range at ± 1 standard deviation as tested with different images, further illuminating the fact that the variation at low filter order is higher than that at high filter order. These results again are helpful for us to explore what parameters we need to adjust for custom wavelet bases.

4 Conclusions

In this paper, we have discussed the theory of using a generalised Gaussian distribution to model the wavelet coefficients within a sub-band, and the implementation to extract the parameters of the distribution from the empirical data. From this, we find out how they vary with decomposition levels and wavelet bases, which can be very useful for us to improve image quality in JPEG 2000.

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