On the minimum stable commutation time for switching nonlinear systems

Graziano Chesi, Senior Member, IEEE

Abstract—Real systems are often driven by switching reference signals which affect dynamics and/or equilibrium points. This paper addresses the computation of upper bounds of the minimum commutation time ensuring stability for switching nonlinear systems. Specifically, we consider the cases of constant and variable equilibrium point of interest, for polynomial systems and for a class of non-polynomial systems. We hence propose upper bounds of the sought minimum commutation time by adopting homogeneous polynomial Lyapunov functions for the former case and polynomial Lyapunov functions for the latter one, which can be computed via LMI optimizations for given Lyapunov functions.

Index Terms—Nonlinear systems, Switching systems, Commutation time, Stability, LMI.

I. INTRODUCTION

Robust analysis of uncertain systems subject to time-varying parametric uncertainties is an important problem since long time. In particular, it has been extensively addressed in the framework of linear systems assuming that the uncertain parameters affect linearly the state space matrix and belong to a given polytope. The various contributions can be divided depending on the assumptions made on the variation rate of the uncertain parameters. Specifically, for the case of unbounded variation rate, see for example [1]–[3] where the use of common quadratic Lyapunov functions, polyhedral Lyapunov functions, and piecewise quadratic Lyapunov functions have been respectively proposed. In the case of null variation rate, corresponding to time-invariant parametric uncertainty, parameter-dependent Lyapunov functions have been successfully employed, see for example [4]–[7] where linear and polynomial dependence on the uncertain parameters have been introduced. Lastly, the case when the variation rate is bounded by a known quantity has been considered, and approaches such as [8], [9] based on affine parameter-dependent quadratic Lyapunov functions and [10] based on homogeneous parameter-dependent homogeneous polynomial Lyapunov functions have been proposed.

An important class of uncertain systems is the class of switching systems, e.g. systems where the uncertain parameters belong to a given finite set. For each constant value of the uncertain parameters in this set the system is assumed to be stable, and the problem consists of analyzing stability with respect to all possible switching sequences of the uncertain parameters. See for example [11]–[14] and references therein.

As it often happens in real systems, the switches occur due to the fact that the system is driven by switching reference signals which affect dynamics and/or equilibrium points. Two important cases arise. The first case concerns the presence of a constant equilibrium point of interest among all possible switches, and the problem consists of ensuring convergence to this equilibrium for all infinite switching sequences. The second case concerns the presence of a variable equilibrium point of interest, and the problem consists of ensuring convergence to the final equilibrium for all finite switching sequences. In both cases, the convergence depends on the minimum commutation time of the switching sequence, and this paper addresses the computation of upper bounds of the minimum commutation time ensuring stability in these two cases.

Specifically, we consider switching nonlinear systems where the initial condition is uncertain and supposed to belong to a given set. We propose upper bounds of the sought minimum commutation time by adopting homogeneous polynomial Lyapunov functions for the case of constant equilibrium point, and polynomial Lyapunov functions for the case of variable equilibrium point. For switching polynomial systems and given Lyapunov functions these upper bounds can be computed via LMI optimizations by exploiting either sum-of-squares (SOS) relaxations or moments relaxations. Then, the proposed strategy is extended to deal with a class of switching non-polynomial systems by introducing suitable worst-case polynomial approximations of the non-polynomial parts, whose conservatism can be decreased by increasing the degree of the approximations and which does not require undesirable state augmentations. Some examples are presented in order to illustrate the application of the proposed results to real physical systems. A preliminary version of this paper appeared in [15].

The paper is organized as follows. Section II introduces the problem formulation and some preliminaries. Section III describes the proposed results. Section IV presents some examples. Lastly, Section V reports some final remarks.

II. PRELIMINARIES

A. Problem formulation

The notation used throughout the paper is as follows: \( \mathbb{N}, \mathbb{R} \): natural and real numbers sets; \( \nabla v(x) \): first derivative row vector of the function \( v(x) \); \( 0_n \): origin of \( \mathbb{R}^n \); \( I \): identity matrix; \( A^t \): transpose of \( A \); \( A > 0 \) \((A \geq 0)\): symmetric positive definite (semidefinite) matrix \( A \); w.r.t.: with respect to; s.t.: subject to.

Let us consider the nonlinear system

\[
\begin{align*}
\dot{x}(t) &= f(x(t), r(t)) + f_{NP}(x(t), r(t)) \\
x(0) &\in \mathcal{A} \\
r(t) &= b_{si(t)} \quad \forall t \in [t_i, t_{i+1}), \\
&\quad i \geq 1, \ t_1 = 0, \ t_i < t_{i+1}
\end{align*}
\]

(1)

G. Chesi is with the Department of Electrical and Electronic Engineering, University of Hong Kong. Contact information: see http://www.eee.hku.hk/~chesi
where \( x(t) \in \mathbb{R}^n \) is the state, \( \mathcal{A} \subset \mathbb{R}^n \) is a given set of possible initial conditions, \( r(t) \in \mathbb{R}^n \) is a piecewise constant reference, \( b_1, \ldots, b_n \) are given possible values for the input, \( t_1, t_2, \ldots \) are times where the switches occur, and \( s : \mathbb{N} \rightarrow \{1, \ldots, n_0 \} \) is a function assigning the value to the input for each switch.

The functions \( f(x, r) \) and \( f_{NP}(x, r) \) are respectively polynomial and non-polynomial in \( x \). It is assumed that \( f_{NP}(x, r) \) has the structure considered in (36). No assumption is made on the dependence of \( f(x, r) \) and \( f_{NP}(x, r) \) on \( r \). The set \( \mathcal{A} \) is supposed defined by

\[
\mathcal{A} = \{ x \in \mathbb{R}^n : a_i(x) \geq 0 \quad \forall i = 1, \ldots, n_0 \} \tag{2}
\]

where \( a_1(x), \ldots, a_n(x) \in \mathbb{R}^{n_a} \) are polynomials (as it will become clear in the sequel, this choice allows one to tackle the conditions proposed in this paper via LMI optimizations).

Let \( c_i \in \mathbb{R}^n \) be the equilibrium point of interest corresponding to \( r(t) = b_i \), hence satisfying

\[
f(c_i, b_i) + f_{NP}(c_i, b_i) = 0. \tag{3}
\]

The minimum commutation time of the switching sequence \( r(t) \) in (1) is defined as

\[
\tau = \inf_{i \geq 1} (t_{i+1} - t_i). \tag{4}
\]

We address the following problems.

(P1) (Constant equilibrium point of interest). Supposing a constant equilibrium point \( c_i = c \) for all \( i = 1, \ldots, n_0 \), compute an upper bound of the minimum commutation time ensuring convergence to \( c \) for infinite switching sequences \( r(t_1), r(t_2), \ldots \) for all initial conditions \( x(0) \) in the set \( \mathcal{A} \):

\[
\tau_{1_{\text{min}}} = \inf \{ t_{i+1} - t_i, \ i \geq 1 : \lim_{t \to +\infty} x(t) = c \ \\
\forall x(\cdot) \text{ satisfying (1)} \}; \tag{5}
\]

(P2) (Variable equilibrium point of interest). Compute an upper bound of the minimum commutation time ensuring convergence to the final equilibrium point \( c_{a(j)} \) for finite switching sequences \( r(t_1), \ldots, r(t_{\kappa}) \) for all initial conditions \( x(0) \) in the set \( \mathcal{A} \):

\[
\tau_{2_{\text{min}}} = \inf \{ t_{i+1} - t_i, \ i = 1, \ldots, \kappa - 1 : \lim_{t \to +\infty} x(t) = c_{s(\kappa)} \ \\
\forall x(\cdot) \text{ satisfying (1)} \}. \tag{6}
\]

Problems P1 and P2 will be addressed in Sections III-A–III-B for polynomial systems, and in Section III-C for a class of non-polynomial systems.

B. SOS and moments relaxations

SOS relaxations can be formulated via LMI optimizations by using the square matricial representation (SMR) [16], [17], known also as Gram matrix [18]. Specifically, let \( p(x) \) be a polynomial of degree \( 2m \), and let \( x^{(m)} \in \mathbb{R}^{\nu(n,m)} \) be a vector containing all monomials of degree less than or equal to \( m \) in \( x \). The SMR of \( p(x) \) w.r.t \( x^{(m)} \) is defined as

\[
p(x) = x^{(m)'} P(\alpha) x^{(m)} \tag{7}
\]

\[
P(\alpha) = P + N(\alpha) \tag{8}
\]

where \( P \) is any symmetric matrix such that \( p(x) = x^{(m)'} P(\alpha) x^{(m)} \), \( \alpha \in \mathbb{R}^{\nu(n,m)} \) is a vector of free parameters, and \( N(\alpha) \) is a linear parametrization of the set

\[
N(m) = \{ N = N' : x^{(m)'} N x^{(m)} = 0 \}. \tag{9}
\]

The dimensions \( \nu(n, m) \) and \( \nu(n, m) \) are given by

\[
\nu(n, m) = \frac{(n + m)!}{n! m!}, \tag{10}
\]

\[
\sigma(n, m) = \frac{1}{2} \nu(n, m)(\sigma(n, m) + 1) - \sigma(n, 2m). \tag{11}
\]

Polynomials with special structures such as homogeneous polynomials can be represented by using a more compact SMR, in particular through a suitably reduced vector \( x^{(m)} \).

The condition \( p(x) \) is non-negative can be relaxed as \( p(x) \) is a SOS, and this latter condition holds if and only if there exists \( \alpha \) such that \( P(\alpha) \geq 0 \), which is an LMI in the variable \( \alpha \). This was proposed in [16], [17] together with the parametrization \( P(\alpha) \). See also [19] where the conservatism of this condition is investigated.

Also, given two polynomials \( q_1(x) \) and \( q_2(x) \), the condition

\[
q_1(x) \geq 0 \text{ \ } \forall x : \ q_2(x) \geq 0 \tag{12}
\]

can be relaxed via the Positivstellensatz [20] as

\[
\exists p_1(x) \text{ such that } p_1(x) \text{ and } p_2(x) = q_1(x) - p_1(x)q_2(x) \text{ are SOS } \tag{13}
\]

and this latter condition holds if and only if the system of LMIs \( P_1(\alpha_1) \geq 0 \) and \( P_2(\alpha_1, \alpha_2) \geq 0 \) is fulfilled, where the complete SMR matrix \( P_2(\alpha_1, \alpha_2) \) of \( p_2(x) \) depends affinely on the coefficients of \( p_1(x) \).

These non-negativity conditions can be investigated via analogous LMI optimizations by adopting the moment relaxations proposed in [21]. See also the MATLAB toolboxes GloptiPoly [22], SOSTOOLS [23] and YALMIP [24] where these LMI optimizations are implemented.

III. PROPOSED RESULTS

A. Constant equilibrium point for polynomial systems

Let us consider problem P1 for polynomial systems, e.g. with \( f_{NP}(x(t), r(t)) = 0_n \). In order to do this, we propose the use of Lyapunov functions in the class of homogeneous polynomials. In particular, for all \( i = 1, \ldots, n_0 \) let \( v_i(x) \) be a polynomial such that

\[
v_i(x) \text{ is radially unbounded and positive definite } \tag{15}
\]

\[
\nabla v_i(x) f(x, b_i) \text{ is locally negative definite } \tag{16}
\]

\[
v_i(\delta x) = \delta^{2m} v_i(x) \forall \delta \in \mathbb{R} \forall x, \ m \geq 1. \tag{17}
\]
Let us denote with $V_i(d)$ the sublevel set

$$V_i(d) = \{ x \in \mathbb{R}^n : v_i(x) \leq d \}. \tag{18}$$

**Theorem 1:** Let $v_1(x), \ldots, v_{n_b}(x)$ be given polynomials satisfying conditions (15)–(17) and

(C1) $A \subseteq V_i(1)$ for all $i = 1, \ldots, n_b$;

(C2) $V_i(\gamma) \subseteq V_j(1)$ for all $i, j = 1, \ldots, n_b$, $i \neq j$;

(C3) $\nabla v_i(x)f(x, b_i) \leq -\omega v_i(x)$ for all $x \in V_i(1) \setminus \{0_n\}$ for all $i = 1, \ldots, n_b$,

where $\lambda \in \mathbb{N}$ belongs to $[1, n_b]$, and $\gamma, \omega \in \mathbb{R}$ are positive.

Then, for all infinite switching sequences $r(t_1), r(t_2), \ldots$ with minimum commutation time $\tau$ greater than

$$\tau_1 = \frac{1}{\omega} \ln \frac{1}{\gamma} \tag{19}$$

the state converges to $0_n$ for all $x(0) \in A$.

**Proof** From C3 one has that $V_i(1)$ is an invariant set for (1) with $r(t) = b_i$. Moreover,

$$r(\bar{t}) = b_i \ \forall \bar{t} \in [t, t + \zeta] \tag{20}$$

$$v_i(x(t + \zeta)) \leq e^{-\omega \zeta} v_i(x(t)) \ \forall \zeta \in \mathbb{R}, \ \zeta \geq 0$$

and

$$r(\bar{t}) = b_i \ \forall \bar{t} \in [t, t + \zeta]$$

$$x(t + \zeta) \in V_i(\gamma \delta) \ \forall \zeta \in \mathbb{R}, \ \zeta \geq \tau$$

where

$$\delta = \gamma \tau^{-1}. \tag{22}$$

Since each Lyapunov function $v_i(x)$ is homogeneous according to (17), one can write for all $i, j = 1, \ldots, n_b$ and positive $z_0, z_i, z_j \in \mathbb{R}$ that

$$V_i(z_i) \subseteq V_j(z_j) \Leftrightarrow V_i(z_0 z_i) \subseteq V_j(z_0 z_j). \tag{23}$$

Let $x(0)$ be any initial condition in $A$. Since $t_{i+1} - t_i \geq \tau$ for all $i \geq 1$, it follows from C1, C2, (21) and (23) that

$$x(t_1) \in V_{s_1}(1) \downarrow$$

$$x(t_2) \in V_{s_1}(\gamma \delta) \downarrow$$

$$x(t_2) \in V_{s_2}(\delta) \downarrow$$

$$x(t_3) \in V_{s_2}(\gamma \delta^2) \downarrow$$

and hence

$$x(t_k) \in V_{s_k}(\delta^{k-1}). \tag{25}$$

Moreover,

$$\delta^{i-1} = \gamma^{(\tau - 1)(i-1)} = \epsilon^{-\omega(\tau - \tau_1)(i-1)}. \tag{26}$$

Therefore, from (25), (26) and (20) one has

$$x(t) \in V_{s(i)}(e^{\delta t}) \ \forall t \in [t_i, t_{i+1}] \tag{27}$$

$$d(t) = -\omega(\tau - \tau_1)(i-1) - \omega(t - t_i) \tag{28}$$

and hence the theorem holds. Indeed, $V_{s(i)}(e^{\delta t})$ converges to the origin for both infinite sequences (since in this case $i \to \infty$ as $t \to \infty$ and we have supposed $\tau > \tau_1$) and finite sequences (since in this case $t \to t_\infty$ as $t \to \infty$, being $t_\infty$ the last switching time).

**Theorem 1** provides a strategy for establishing an upper bound of $\tau^m_{1, \text{min}}$. Figure 1a illustrates this strategy for the case $n_b = 2$. Let us observe that the essential property (23) is guaranteed by the fact that the polynomial Lyapunov functions we have adopted are homogeneous.

Conditions C1–C3 can be tackled through LMI optimizations as explained in Section II-B. Also, the computation of the least conservative $\gamma$ and $\beta$ in C2–C3 can be handled by these LMI optimizations because the SMR matrices depend affine linearly on these unknowns that, hence, become LMI variables.

In order to answer to the question “When are these LMI solvable?”, let us observe that feasible Lyapunov functions $v_1(x), \ldots, v_{n_b}(x)$ can be simply obtained under the mild assumption that the linearized system is asymptotically stable for each constant input $r(t) = b_i$. Indeed, in this case $v_i(x)$ can be built by solving the Lyapunov equation corresponding to the linearized system for the constant input $r(t) = b_i$, taking the $m$th power of the so found quadratic Lyapunov functions, and then scaling the so constructed Lyapunov functions in order to fulfill condition C1, where $\mathcal{A}$ is any small enough neighborhood of $0_n$. The so built initial guess can be used in order to search for Lyapunov functions admitting an arbitrary given set $\mathcal{A}$ and/or providing less conservative upper bounds of $\tau^m_{1, \text{min}}$, for example via BMI optimizations.

**B. Variable equilibrium point for polynomial systems**

Let us consider problem P2 for polynomial systems, e.g. with $f_{NP}(x(t), r(t)) = 0_n$. In order to $\tau^m_{2, \text{min}}$ exist, it is necessary to suppose that each equilibrium point $c_i$:

1) is locally stable;

2) lies on the domain of attraction of all the other equilibrium points:

$$c_i \in D_j \ \forall i, j = 1, \ldots, n_b, \tag{29}$$

where $D_i \subseteq \mathbb{R}^n$ is the domain of attraction of $c_i$:

$$D_i = \left\{ x(0) \in \mathbb{R}^n : \lim_{t \to +\infty} x(t) \bigg|_{r(t)=b_i} = c_i \right\}. \tag{30}$$

However, these conditions are not sufficient for the stability of the switching system, see for instance Example 2 in Section IV. Indeed, the switching system is stable if and only if $x(t)$ lies on the domain of attraction of the equilibrium point at time $t$ for all $t$.

For all $i = 1, \ldots, n_b$ let $v_i(x)$ be a polynomial such that

$$v_i(x + c_i)$$

is radially unbounded and positive definite (31)

$$\nabla v_i(x)f(x, b_i)|_{x=c+x+c_i}$$

is locally negative definite. (32)

The following result shows how this latter condition can be guaranteed and, consequently, how $\tau^m_{2, \text{min}}$ can be estimated.

**Theorem 2:** Let $v_1(x), \ldots, v_{n_b}(x)$ be given polynomials satisfying conditions (31)–(32), condition C1, and

(C4) $V_i(\gamma_i) \subseteq V_j(1) \ \forall i, j = 1, \ldots, n_b$, $i \neq j$;

(C5) $\nabla v_i(x)f(x, b_i) \leq -\omega_i v_i(x) \ \forall x \in V_i(1) \setminus \{c_i\} \ \forall i = 1, \ldots, n_b$. 

\[ \square \]
where $\gamma_i, \omega_i \in \mathbb{R}$ are positive. Then, for all finite switching sequences $r(t_1), \ldots, r(t_n)$ with minimum commutation time $\tau$ greater than or equal to

$$
\tau_2 = \max_{i=1,\ldots,n} \frac{1}{\omega_i} \ln \frac{1}{\gamma_i},
$$

(33)
the state converges to $c_{\tau(x)}$ for all $x(0) \in \mathcal{A}$.

**Proof.** Analogous to the proof of Theorem 1 one has from C5

$$
r(t) = b_i \quad \forall i \in [t, t + \zeta] \quad \text{and} \quad x(t) \in \mathcal{V}_i(1)
$$

(34)
Consider any $x(0) \in \mathcal{A}$. Since $t_{i+1} - t_i \geq \tau$ for all $i \geq 1$ and we have supposed $\tau \geq \tau_2$, it follows from C1, C4 and (34)

$$
x(t_1) \in \mathcal{V}_{s(1)}(1) \quad \Rightarrow \quad x(t_2) \in \mathcal{V}_{s(2)}(1) \quad \Rightarrow \quad x(t_3) \in \mathcal{V}_{s(3)}(1)
$$

(35)

Figure 1b illustrates the strategy proposed in Theorem 2 for $n_b = 2$. Let us observe that the computation of the least conservative $\gamma_i$ and $\beta_i$ satisfying conditions C4–C5 can be formulated via LMI optimizations as explained in Section II-B.

Let us also observe that condition C5 is unnecessary for the values of $j$ such that $c_j$ is a globally asymptotically stable equilibrium point. Also, let us observe that $\tau_{2,\text{min}} = 0$ if all equilibria are globally asymptotically stable.

### C. Constant and variable equilibrium point for non-polynomial systems

In this section we consider problems P1 and P2 in the case of non-polynomial system (1). In particular, it is assumed that the function $f_{NP}(x(t), r(t))$ has the form

$$
f_{NP}(x(t), r(t)) = \sum_{j=1}^{n_f} g_j(x(t), r(t))h_j(x_j(t), r(t))
$$

(36)
where $n_f \leq n$, $g_j(x(t), r(t))$ is polynomial in $x(t)$, and $h_j(x_j(t), r(t))$ depends on the $j$-th component of $x(t)$ and is non-polynomial. The idea for addressing this case extends the strategy proposed in [25] and consists of representing non-polynomial terms via truncated Taylor expansion centered at the equilibrium point and taking into account the worst-case remainder.

Let us consider first the case of constant equilibrium, and let us write

$$
h_j(x_j(t), r(t)) = \mu_j(x_j, r(t)) + \varrho_j(x_j)\xi_j(\phi, r(t))
$$

(37)
where $\mu_j(x_j, r(t))$ is the truncated expansion and $\varrho_j(x_j)\xi_j(\phi, r(t))$ is the remainder in the Lagrange form, where $\varrho_j(x_j)$ is a monomial and $\xi_j(\phi, r(t))$ is the highest order derivative of $h_j(x_j(t), r(t))$ evaluated at a suitable point $\phi$. Let $\xi_{j,i,-}, \xi_{j,i,+} \in \mathbb{R}$ satisfy

$$
\xi_{j,i,-} = \xi_{j,i,+} \quad \forall \phi \in [\phi_{j,i,-}, \phi_{j,i,+}]
$$

(38)
for all $i = 1, \ldots, n_b$ and $j = 1, \ldots, n_f$.

**Theorem 3:** Let us replace condition C4 with

$$
\nabla v_i(x)(f(x, b_i)) + \sum_{j=1}^{n_f} g_j(x, b_i)(\mu_j(x_j) + \varrho_j(x_j))z_j(x, b_i) \leq -\omega v_i(x)
$$

(39)
for all $(z_1, \ldots, z_{n_f}) \in \{\xi_{j,i,-}, \xi_{j,i,+}\}$ and $i = 1, \ldots, n_b$.

Then, Theorem 1 holds with $g(x)$ as in (36).

**Proof.** Obvious, since from the property of remainder in the Lagrange form we have that

$$
\forall x \in \mathcal{V}_i(1) \exists z_j \in [\xi_{j,i,-}, \xi_{j,i,+}]: h_j(x_j(t), r(t)) = \mu_j(x_j, r(t)) + \varrho_j(x_j)z_j
$$

(40)
and since $\nabla v_i(x)(f(x, b_i)) + \sum_{j=1}^{n_f} g_j(x, b_i)(\mu_j(x_j) + \varrho_j(x_j))z_j(x, b_i)$ is affine linear in $(z_1, \ldots, z_{n_f})$.

The strategy of Theorem 3 consists of taking into account the worst-case remainder of the truncated Taylor expansion of $f_{NP}(x(t), r(t))$ through the quantities $\xi_{j,i,-}, \xi_{j,i,+}$. A result analogous to Theorem 3 can be derived also for Theorem 2.

### IV. Examples

#### A. Example 1

Let us consider the inverted pendulum system in Figure 2a

\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= g l^{-1} \sin x_1(t) - k_1 (ml)^{-2} x_1(t) (1 + k_2 x_1(t)^2) - f m^{-2} x_2(t) + (ml)^{-2} u(t) \\
y(t) &= x_1(t)
\end{align*}

\[\text{(41)}\]
where \( x_1(t) = \theta(t), \ x_2(t) = \dot{\theta}(t), \ m = 0.05 \, \text{Kg} \) is the mass of the bob, \( l = 1 \, \text{m} \) is the length of the rod, \( g \) is the gravity acceleration, \( k_1 = 0.3 \, \text{J} \) and \( k_2 = 0.5 \, \text{rad}^{-2} \) are the coefficients of the hardening spring, \( f = 0.2 \, \text{N-rad-m}^{-1} \) is the coefficient of friction, and \( u(t) \) is the control torque

\[
u(t) = r(t)y(t), \quad r(t) \in \{b_1, b_2\}
\]

where \( r(t) \) is the switching control gain with \( b_1 = -5 \, \text{J} \) and \( b_2 = -0.8 \, \text{J} \). The equilibrium point of interest is the origin, and the set of initial conditions is \( A = \{x : x_1^2 + x_2^2 \leq 0.5^2\} \). The problem is to compute an upper bound of \( \tau_1^{\text{max}} \) in (5). As shown in Figure 2b, the trajectory may diverge through an infinite sequence of switches.

Let us use Theorem 3. We decompose \( \sin x_1 \) as in (37) selecting a truncated expansion of degree 2 and a remainder of degree 3. Since the third derivative of \( \sin x_1 \) is \( -\cos x_1 \), it follows that we can simply select \( \xi_{j,i,-} = -1 \) and \( \xi_{j,i,+} = 1 \) in order to fulfill (38). By proceeding as described at the end of Section III-A we find the upper bounds \( \tau_1 = 1.40 \, \text{s} \) obtained via quadratic Lyapunov functions, and \( \tau_1 = 1.02 \, \text{s} \) obtained via homogeneous polynomial Lyapunov functions of degree 4. Tighter upper bounds can be found by increasing the degree of the Lyapunov functions and Taylor expansion of \( \sin x_1 \).

\[ \begin{cases} \dot{x}_1(q) = x_2(q) \\ \dot{x}_2(q) = -x_1(q) - \sqrt{L/C} x_2(q) (1 - x_1(q)^2) + r(q) \end{cases} \]

where \( q = (LC)^{-\frac{1}{2}} t \) is the scaled time variable, \( x_1(q) = v_C(q), \ x_2(q) = \dot{v}_C(q), \ L = 0.01 \, \text{H}, \ C = 0.08 \, \text{F}, \) the sub-circuit enclosed in the dashed box has current-voltage relation \( i = h(v) \) with \( h(v) = v - \frac{1}{3} v^3 \) [26], and \( r(q) \) is the switching reference voltage

\[
r(q) \in \{b_1, b_2\}
\]

where \( b_1 = 0 \, \text{V} \) and \( b_2 = 0.5 \, \text{V} \). The equilibria of interest are \( c_1 = 0_2 \) and \( c_2 = (0.5, 0)^T \), and the set of initial conditions is \( A = \{x : x_1^2 + x_2^2 \leq 0.8^2\} \). The problem is to compute an upper bound of \( \tau_2^{\text{max}} \) in (6). As shown in Figure 3b, the trajectory can diverge through a finite sequence of switches.

By using Theorem 2 we find the upper bounds \( \tau_2 = 0.619 \, \text{s} \) for quadratic Lyapunov functions, and \( \tau_2 = 0.427 \, \text{s} \) for polynomial Lyapunov functions of degree 4.

\[ \begin{cases} \dot{x}_1(q) = x_2(q) \\ \dot{x}_2(q) = -x_1(q) - \sqrt{L/C} x_2(q) (1 - x_1(q)^2) + r(q) \end{cases} \]

where \( q = (LC)^{-\frac{1}{2}} t \) is the scaled time variable, \( x_1(q) = v_C(q), \ x_2(q) = \dot{v}_C(q), \ L = 0.01 \, \text{H}, \ C = 0.08 \, \text{F}, \) the sub-circuit enclosed in the dashed box has current-voltage relation \( i = h(v) \) with \( h(v) = v - \frac{1}{3} v^3 \) [26], and \( r(q) \) is the switching reference voltage

\[
r(q) \in \{b_1, b_2\}
\]

where \( b_1 = 0 \, \text{V} \) and \( b_2 = 0.5 \, \text{V} \). The equilibria of interest are \( c_1 = 0_2 \) and \( c_2 = (0.5, 0)^T \), and the set of initial conditions is \( A = \{x : x_1^2 + x_2^2 \leq 0.8^2\} \). The problem is to compute an upper bound of \( \tau_2^{\text{max}} \) in (6). As shown in Figure 3b, the trajectory can diverge through a finite sequence of switches.

By using Theorem 2 we find the upper bounds \( \tau_2 = 0.619 \, \text{s} \) for quadratic Lyapunov functions, and \( \tau_2 = 0.427 \, \text{s} \) for polynomial Lyapunov functions of degree 4.

\[ \begin{cases} \dot{x}_1(q) = x_2(q) \\ \dot{x}_2(q) = -x_1(q) - \sqrt{L/C} x_2(q) (1 - x_1(q)^2) + r(q) \end{cases} \]

where \( q = (LC)^{-\frac{1}{2}} t \) is the scaled time variable, \( x_1(q) = v_C(q), \ x_2(q) = \dot{v}_C(q), \ L = 0.01 \, \text{H}, \ C = 0.08 \, \text{F}, \) the sub-circuit enclosed in the dashed box has current-voltage relation \( i = h(v) \) with \( h(v) = v - \frac{1}{3} v^3 \) [26], and \( r(q) \) is the switching reference voltage

\[
r(q) \in \{b_1, b_2\}
\]

where \( b_1 = 0 \, \text{V} \) and \( b_2 = 0.5 \, \text{V} \). The equilibria of interest are \( c_1 = 0_2 \) and \( c_2 = (0.5, 0)^T \), and the set of initial conditions is \( A = \{x : x_1^2 + x_2^2 \leq 0.8^2\} \). The problem is to compute an upper bound of \( \tau_2^{\text{max}} \) in (6). As shown in Figure 3b, the trajectory can diverge through a finite sequence of switches.

By using Theorem 2 we find the upper bounds \( \tau_2 = 0.619 \, \text{s} \) for quadratic Lyapunov functions, and \( \tau_2 = 0.427 \, \text{s} \) for polynomial Lyapunov functions of degree 4.

\[ \begin{cases} \dot{x}_1(q) = x_2(q) \\ \dot{x}_2(q) = -x_1(q) - \sqrt{L/C} x_2(q) (1 - x_1(q)^2) + r(q) \end{cases} \]

where \( q = (LC)^{-\frac{1}{2}} t \) is the scaled time variable, \( x_1(q) = v_C(q), \ x_2(q) = \dot{v}_C(q), \ L = 0.01 \, \text{H}, \ C = 0.08 \, \text{F}, \) the sub-circuit enclosed in the dashed box has current-voltage relation \( i = h(v) \) with \( h(v) = v - \frac{1}{3} v^3 \) [26], and \( r(q) \) is the switching reference voltage

\[
r(q) \in \{b_1, b_2\}
\]

where \( b_1 = 0 \, \text{V} \) and \( b_2 = 0.5 \, \text{V} \). The equilibria of interest are \( c_1 = 0_2 \) and \( c_2 = (0.5, 0)^T \), and the set of initial conditions is \( A = \{x : x_1^2 + x_2^2 \leq 0.8^2\} \). The problem is to compute an upper bound of \( \tau_2^{\text{max}} \) in (6). As shown in Figure 3b, the trajectory can diverge through a finite sequence of switches.

By using Theorem 2 we find the upper bounds \( \tau_2 = 0.619 \, \text{s} \) for quadratic Lyapunov functions, and \( \tau_2 = 0.427 \, \text{s} \) for polynomial Lyapunov functions of degree 4.
We consider the switching among the following two static system with 5 states, 2 inputs and 4 outputs described by

\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0.154 & 0.0042 & 1.54 & 0 \\
0 & 0.249 & 1 & 5.2 & 0 \\
0.0386 & 0.996 & 0.0003 & 0.117 & 0 \\
0 & 0.5 & 0 & 0 & 0.5
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
0 & 0 & 0.744 & 0.032 \\
0.337 & 1.12 & 0.02 & 0 \\
0 & 0 \\
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]

We consider the switching among the following two static output feedback controllers:

\[
K_1 = \begin{pmatrix}
-0.156 & -4.74 & -20.4 & -27.9 \\
-2.03 & -5.34 & -5.02 & -0.914 \\
\end{pmatrix},
\]

\[
K_2 = \begin{pmatrix}
2.33 & -2.57 & -19.8 & -23.2 \\
-2.36 & -4.95 & -5.25 & -1.04 \\
\end{pmatrix}.
\]

Lastly, in order to consider a more difficult example, we make this system nonlinear by introducing some nonlinearities:

\[
\dot{x} = (A + BK_1C)x + 0.05 \left( x_3^3, x_2^3, x_4^3, x_5^3 \right), \quad i = 1, 2.
\]

It can be observed also that these nonlinearities, in spite of their simple form, tend to destabilize the equilibrium, indeed \( x(t) \) diverges for all initial conditions \( x(0) \) with sufficiently large magnitude. As a consequence, the domain of attraction of the origin is bounded, and hence it can be expected that the estimation of \( \tau_{\min}^1 \) is more involved than with nonlinearities of the same degree for which this domain is unbounded, more specifically it can be expected that verifying condition (C3) in Theorem 1 is more difficult.

This system is in the form of (1) with \( r(t) \) containing the coefficients of the controller, which switches between \( K_1 \) and \( K_2 \). We choose \( \mathcal{A} = \{ x : \sum_{i=1}^{5} x_i^2 \leq 0.012 \} \).

It is worthwhile to note that in this example computing upper bounds of \( \tau_{\min}^1 \) is a difficult task by using non-LMI techniques even in the simplest case that the considered Lyapunov functions are quadratic. Indeed, conditions C1–C3 involve nonconvex constrained optimizations, in particular for C3 one has to establish whether polynomials of degree 4 in 5 variables are non-positive inside the sublevel sets \( V_i(1) \). Instead, Theorem 1 provides a systematic way not only to verify conditions C1–C3 but also for computing less conservative \( \omega, \gamma \) satisfying these conditions, in particular providing the upper bound \( \tau_1 = 93.8 \).

V. CONCLUSION

This paper has addressed the computation of upper bounds of the minimum commutation time ensuring stability for switching nonlinear systems for constant and variable equilibrium points, by proposing the use of homogeneous polynomial Lyapunov functions for the former case and polynomial Lyapunov functions for the latter one. It has been shown that the proposed strategy can be applied to polynomial systems as well as non-polynomial systems by introducing worst-case truncations of the non-polynomial parts.

ACKNOWLEDGEMENT

The author thanks the Associate Editor and Reviewers.

REFERENCES