Robust Static Output Feedback Controllers via Robust Stabilizability Functions

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Abstract

This paper addresses the design of robust static output feedback controllers that minimize a polynomial cost and robustly stabilize a system with polynomial dependence on an uncertain vector constrained in a semialgebraic set. The admissible controllers are those in a given hyper-rectangle for which the system is well-posed. First, the class of robust stabilizability functions is introduced, i.e., the functions of the controller that are positive whenever the controller robustly stabilizes the system. Second, the approximation of a robust stabilizability function with a controller-dependent lower bound is proposed through a sums-of-squares (SOS) program exploiting a technique developed in the estimation of the domain of attraction. Third, the derivation of a robust stabilizing controller from the found controller-dependent lower bound is addressed through a second SOS program that provides an upper bound of the optimal cost. The proposed method is asymptotically non-conservative under mild assumptions.

Index Terms

Uncertain system, Robust control, Robust stabilizability function, SOS polynomial.

I. INTRODUCTION

A key problem in systems with uncertainty consists of designing robust stabilizing controllers, in particular feedback controllers that, without requiring to measure the uncertainty, ensure robust stability (i.e., stability for all admissible uncertainties) of the closed-loop system. Numerous approaches have been proposed for robust stability analysis of systems affected by parametric uncertainties, mainly based on the use of Lyapunov functions and convex optimization problems with linear matrix inequalities (LMIs), see e.g. [2], [6], [8], [9], [11], [15]. Unfortunately, these

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approaches unavoidably lead to nonconvex optimization whenever applied to robust control design. In fact, whenever a controller to be designed is present, the LMIs generally become bilinear matrix inequalities (BMIs) in the unknown Lyapunov function and controller. In order to cope with this issue, several approaches have been proposed, for instance based on the introduction of generalized multipliers and slack variables. Although conservative, these approaches are quite flexible as they allow one to cope with several performance requirements such as minimization of the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms. See e.g. [1], [7], [13]. Other approaches have been proposed without the use of Lyapunov functions, such as [6] which provides conditions for robust stability, and [10] which estimates robust stability regions.

This paper addresses the design of robust static output feedback controllers that minimize a polynomial cost and robustly stabilize a system with polynomial dependence on an uncertain vector constrained in a semialgebraic set. The admissible controllers are those in a given hyper-rectangle for which the system is well-posed. First, the class of robust stabilizability functions is introduced, i.e., the functions of the controller that are positive whenever the controller robustly stabilizes the system. Second, the approximation of a robust stabilizability function with a controller-dependent lower bound is proposed through a SOS program exploiting a technique developed in the estimation of the domain of attraction. Third, the derivation of a robust stabilizing controller from the found controller-dependent lower bound is addressed through a second SOS program that provides an upper bound of the optimal cost. The proposed method is asymptotically non-conservative under mild assumptions. A conference version of this paper (without the proofs and the convergence analysis) will appear as reported in [5].

II. Problem Formulation

Notation: $\mathbb{R}$: real numbers; $A^\prime$: transpose; $\det(A)$: determinant; $\text{adj}(A)$: adjoint; $\text{spec}(A)$: set of eigenvalues; $\text{col}(A)$: column vector stacking the columns of $A$; $A > 0, \, A \geq 0$: symmetric positive definite and symmetric positive semidefinite matrix; Hurwitz matrix: matrix with all eigenvalues having negative real part; $\deg(a)$: degree. Let us consider

$$
\begin{align*}
\dot{x}(t) &= A(p)x(t) + B(p)u(t) \\
y(t) &= C(p)x(t) + D(p)u(t)
\end{align*}
$$

(1)
where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $y(t) \in \mathbb{R}^{n_y}$, $p \in \mathbb{R}^q$, and $A(p)$, $B(p)$, $C(p)$ and $D(p)$ are matrix polynomials. It is supposed that

$$p \in \mathcal{P}$$

$$\mathcal{P} = \{ p \in \mathbb{R}^q : a_i(p) \geq 0, b_j(p) = 0, \ i = 1 \ldots, n_a, \ j = 1, \ldots, n_b \}$$

where $a_i(p)$ and $b_j(p)$ are polynomials. This system is controlled via

$$u(t) = Ky(t)$$

where $K \in \mathbb{R}^{n_u \times n_y}$ is the controller to be determined. It is supposed that

$$K \in \mathcal{K}$$

where $\mathcal{K} \subset \mathbb{R}^{n_u \times n_y}$ is the hyper-rectangle

$$\mathcal{K} = \{ K \in \mathbb{R}^{n_u \times n_y} : k_{ij}^- \leq k_{ij} \leq k_{ij}^+, \ i = 1, \ldots, n_u, \ j = 1, \ldots, n_y \}$$

where $k_{ij} \in \mathbb{R}$ is the $(i,j)$-th entry of $K$, and $k_{ij}^-$, $k_{ij}^+$ are its lower and upper bounds. The closed-loop system (1)–(6) can be rewritten as

$$\begin{cases}
\dot{x}(t) = A_{cl}(K, p)x(t) \\
K \in \mathcal{K} \cap \mathcal{K}_{wp} \\
p \in \mathcal{P}
\end{cases}$$

$$A_{cl}(K, p) = A(p) + B(p)K(I - D(p)K)^{-1}C(p)$$

where $\mathcal{K}_{wp}$ is the set of controllers such that $A_{cl}(K, p)$ is well-posed. In particular, we say that $A_{cl}(K, p)$ is well-posed if

$$| \det(I - D(p)K) | \geq \rho_{wp} \ \forall p \in \mathcal{P},$$

where $\rho_{wp} > 0$ is an arbitrary small chosen threshold. Hence,

$$\mathcal{K}_{wp} = \{ K \in \mathbb{R}^{n_u \times n_y} : (9) \text{ holds} \}.$$  

The system (7) is said robustly stable if, for some $\rho_s \geq 0$,

$$\Re(\lambda) < -\rho_s \ \forall \lambda \in \text{spec}(A_{cl}(K, p)) \ \forall p \in \mathcal{P}.$$  

**Problem.** Establish the existence of a robust stabilizing controller for the system (7), i.e., the non-emptyness of

$$\mathcal{K}_s = \{ K \in \mathcal{K} \cap \mathcal{K}_{wp} : (11) \text{ holds} \}.$$
Also, we aim to determine a controller in $K_s$ that minimizes a given polynomial cost $r(K)$, i.e.,
\[ r^* = \inf_{K \in K_s} r(K). \] (13)
Indeed, since $K_s$ generally contains an infinite number of controllers whenever it is non-empty, one might want to pick one according to some criterion. For instance, this can be the minimization of the Euclidean norm of the entries of $K$, since actuators with small gains can be preferable in real systems as they require less power.

### III. Robust Stabilizability Functions

We say that $s : \mathbb{R}^{n_u \times n_y} \to \mathbb{R}$ is a robust stabilizability function over the set $K \cap K_{wp}$ for the system (7) if and only if
\[ K \in K \cap K_{wp} \Rightarrow \begin{cases}
  s(K) > 0 & \text{if } K \in K_s \\
  s(K) \leq 0 & \text{otherwise}.
\end{cases} \] (14)
Here we show how to build a robust stabilizability function through the use of the Routh-Hurwitz criterion on the characteristic polynomial of the matrix $A_{cl}(K, p)$. Let us express $A_{cl}(K, p)$ as
\[ A_{cl}(K, p) = \frac{\bar{A}_{cl}(K, p)}{\det(I - D(p)K)} \] (15)
where $\bar{A}_{cl}(K, p)$ is the matrix polynomial
\[ \bar{A}_{cl}(K, p) = \det(I - D(p)K)A(p) + B(p)K \text{adj}(I - D(p)K)C(p) \] (16)
and $\text{adj}(I - D(p)K)$ is the adjoint matrix of $I - D(p)K$. In order to get rid of the denominator in (15), we need to consider two possible cases depending on its sign. To this end, let us define
\[ \mathcal{T} = \begin{cases}
  \{0\} & \text{if } D(p) = 0 \\
  \{0, 1\} & \text{otherwise}
\end{cases} \] (17)
and the partition of $K_{wp}$ given by
\[ K_{wp} = \bigcup_{\tau \in \mathcal{T}} \mathcal{U} \] (18)
\[ \mathcal{U} = \{K \in \mathbb{R}^{n_u \times n_y} : (-1)^\tau \det(I - D(p)K) \geq \rho_{wp} \ \forall p \in \mathcal{P}\}. \] (19)
Let $z \in \mathbb{R}^{n_z}$ be the variable
\[ z = \begin{pmatrix} \text{col}(K) \\ p \end{pmatrix}, \quad n_z = n_u n_y + q. \] (20)
For $\tau \in \mathcal{T}$, let us define
\[ W(z) = (-1)^\tau \left( \bar{A}_{cl}(K, p) + \rho_s \det(I - D(p)K)I \right) \] (21)
and its characteristic polynomial
\[ v(\lambda, z) = \det (\lambda I - W(z)) \] (22)
where $\lambda \in \mathbb{C}$. Let us express $v(\lambda, z)$ as
\[ v(\lambda, z) = \lambda^n + \sum_{i=0}^{n-1} d_i(z)\lambda^i \] (23)
where $d_i(z)$ are polynomials. Let us write the Routh-Hurwitz table of $v(\lambda, z)$ as
\[ \psi_i^j(z) = \prod_{l=i-1, i-3, \ldots} \bar{e}_l^0(z), \quad i = 2, \ldots, n, \quad j = 0, 1, \ldots \] (24)
We have that $e_{ij}(z)$ can be expressed as
\[ e_{ij}(z) = \frac{\bar{e}_{ij}(z)}{\hat{e}_{ij}(z)}, \quad \hat{e}_{ij}(z) = \prod_{l=i-1, i-3, \ldots} \bar{e}_l^0(z). \] (25)
where $\bar{e}_{ij}(z)$ and $\hat{e}_{ij}(z)$ are polynomials. Let us define the set
\[ \mathcal{N} = \{ i = 0, \ldots, n : \bar{e}_{i0}(z) \text{ is a non-positive constant} \} \] (26)
and let $f_m(z)$, $m = 1, \ldots, n_f$, be the non-constant polynomials among $\bar{e}_{i0}(z)$, $i = 0, \ldots, n$.

Theorem 1: Let $\tau \in \mathcal{T}$. If $\mathcal{N} \neq \emptyset$, then
\[ (11) \text{ does not hold for any } K \in \mathcal{U}. \] (27)
Hence, suppose that $\mathcal{N} = \emptyset$, and let us define
\[ s(K) = \inf_{p \in \mathcal{P}} \min_{m=1, \ldots, n_f} f_m(z). \] (28)
Then,
\[ \begin{cases} 
    s(K) > 0 \\
    K \in \mathcal{K} \cap \mathcal{U} 
\end{cases} \Rightarrow K \in \mathcal{K}_s. \] (29)
Moreover, if $\mathcal{P}$ is compact, this condition holds in both directions, i.e., $s(K)$ is a robust stabilizability function over the set $\mathcal{K} \cap \mathcal{U}$ for the system (7).
Proof. Let us consider $N = \emptyset$, and suppose that $s(K) > 0$ for $K \in \mathcal{K} \cap \mathcal{U}$. This implies that

$$\bar{e}_{i0}(z) > 0 \; \forall i = 0, \ldots, n \; \forall p \in \mathcal{P}.$$  

From (25) one obtains

$$e_{i0}(z) > 0 \; \forall i = 0, \ldots, n \; \forall p \in \mathcal{P}$$

hence implying that

$W(z)$ is Hurwitz $\forall p \in \mathcal{P}$.

Since $K \in \mathcal{U}$ one has

$$(-1)^r \det(I - D(p)K) \geq \rho_{wp} \; \forall p \in \mathcal{P}$$

and, hence,

$$\frac{W(z)}{(-1)^r \det(I - D(p)K)} \text{ is Hurwitz } \forall p \in \mathcal{P}.$$  

This implies that

$$\frac{\tilde{A}_{sl}(K, p)}{\det(I - D(p)K)} + \rho_s I \text{ is Hurwitz } \forall p \in \mathcal{P}$$

and, hence, that (11) holds, i.e., $K \in \mathcal{K}_s$.

Next, let us consider $N = \emptyset$ and $K \in \mathcal{K}_s$, and suppose that $\mathcal{P}$ is compact. Reversing the previous part of the proof, one shows that

$$f_m(z) > 0 \; \forall m = 1, \ldots, n_f \; \forall p \in \mathcal{P}.$$  

Since $s(K)$ is the infimum of the polynomials $f_m(z)$ over $\mathcal{P}$, and $\mathcal{P}$ is compact, it follows that

$$s(K) > 0$$

and, hence, $s(K)$ is a robust stabilizability function over the set $\mathcal{K} \cap \mathcal{U}$ for the system (7).

Lastly, let us suppose that $N \neq \emptyset$. Let $i$ be the minimum of the set $N$. It follows that

$$\bar{e}_{i0}(z) \leq 0 \; \forall K \in \mathcal{U} \; \forall p \in \mathcal{P}.$$  

This implies that a necessary condition for the existence of $K \in \mathcal{U}$ such that

$$e_{i0}(z) > 0 \; \forall p \in \mathcal{P}$$

is that, for such a $K$,

$$\dot{e}_{i0}(z) < 0 \; \forall p \in \mathcal{P}.$$  

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But this implies from (25) that, for such a $K$, there exists $l < i$ such that

$$e_{l0}(z) < 0$$

for some $p \in \mathcal{P}$. Consequently, there does not exist $K \in \mathcal{U}$ such that the entries of the first column of the Routh-Hurwitz table built for the characteristic polynomial $v(\lambda, z)$ of $W(z)$ are positive for all $p \in \mathcal{P}$. Consequently, (27) holds. $\square$

Theorem 1 states that a robust stabilizability function over the set $\mathcal{K} \cap \mathcal{U}$ for the system (7) can be built using the polynomials $f_m(z)$ whenever $\mathcal{P}$ is compact. If $\mathcal{P}$ is not compact, one can still use the function $s(K)$ to obtain a sufficient condition for identifying robust stabilizing controllers in $\mathcal{K} \cap \mathcal{U}$ according to (29).

IV. CONTROLLER-DEPENDENT LOWER BOUND

Since computing the function $s(K)$ in (28) is not easy, we look for an approximation of it, in particular a controller-dependent lower bound of $s(K)$ since the positivity of such a lower bound will imply the positivity of $s(K)$. Let us denote with $\xi(K)$ the sought controller-dependent lower bound. We consider the case where $\xi(K)$ is a polynomial, and denote with $d_\xi$ its degree. The idea for computing $\xi(K)$ is to exploit the technique we introduced in [3] for estimating the parameter-dependent lower bound of estimates of the robust domain of attraction, where the gap between the original function and the polynomial lower bound is minimized by maximizing the integral of the latter. Let us start by rewriting the set $\mathcal{K} \cap \mathcal{U}$ as

$$\mathcal{K} \cap \mathcal{U} = \{ K \in \mathbb{R}^{n_u \times n_y} : c_l(z) \geq 0 \ \forall p \in \mathcal{P}, \ l = 1, \ldots, n_c \}$$

(30)

where $c_l(z)$ are polynomials. Indeed, in order to impose $K \in \mathcal{K}$, one can define

$$c_l(z) = (k_{ij} - k_{ij}^{-}) (k_{ij}^{+} - k_{ij})$$

(31)

for all possible $i, j$ since, with this choice,

$$c_l(z) \geq 0 \iff k_{ij} \in [k_{ij}^{-}, k_{ij}^{+}].$$

(32)

Moreover, in order to impose $K \in \mathcal{U}$, one can define

$$c_l(z) = c_{l0}(z) ((-1)^r \det(I - D(p)K) - \rho_{wp})$$

(33)
where \( c_{\ell 0}(z) \) is any positive polynomial over \( K \times \mathcal{P} \) since, with this choice and over this set,
\[
c_{\ell}(z) \geq 0 \iff \begin{cases} 
\det(I - D(p)K) \geq \rho_{wp} & \text{if } \tau = 0 \\
\det(I - D(p)K) \leq -\rho_{wp} & \text{if } \tau = 1.
\end{cases}
\] (34)

Let us define \( n_g = n_f \), and, for \( m = 1, \ldots, n_g \), let us define the polynomials
\[
g_m(z) = f_m(z) - \xi(K) - \sum_{i=1}^{n_a} a_i(p)\alpha_{mi}(z) - \sum_{j=1}^{n_b} b_j(p)\beta_{mj}(z) - \sum_{l=1}^{n_c} c_l(z)\gamma_{ml}(z)
\] (35)
where \( \alpha_{mi}(z), \beta_{mj}(z) \) and \( \gamma_{ml}(z) \) are polynomials. Then, let \( h(\xi) \) be the function that satisfies
\[
h(\xi) = \int_{K \in \mathcal{K}} \psi(K)\xi(K)dK
\] (36)
where \( \psi(K) \) is a positive polynomial over \( \mathcal{K} \) (we will discuss the choice of \( \psi(K) \) in the Section VI). Let us observe that \( h(\xi) \) is a linear function on the coefficients of \( \xi(K) \) and can be easily computed since \( \mathcal{K} \) is a hyper-rectangle.

**Theorem 2:** Let \( \tau \in \mathcal{T} \), and suppose that \( \mathcal{N} = \emptyset \). Define the SOS program
\[
\sup_{\xi, \alpha_{mi}, \beta_{mj}, \gamma_{ml}} h(\xi)
\] s.t.
\[
\begin{align*}
&g_m(z) \text{ is SOS} \\
&\alpha_{mi}(z) \text{ is SOS} \\
&\gamma_{ml}(z) \text{ is SOS}
\end{align*}
\] (37)

Let \( \xi^*(K) \) be the optimal value of \( \xi(K) \) in (37). Then,
\[
\xi^*(K) \leq s(K) \quad \forall K \in \mathcal{K} \cap \mathcal{U}
\] (38)
where \( s(K) \) is as in (28). Therefore,
\[
\xi^*(K) > 0 \text{ and } K \in \mathcal{K} \cap \mathcal{U} \implies K \in \mathcal{K}_s.
\] (39)

**Proof.** Let us suppose that the constraints in (37) hold. Let us consider any \( p \in \mathcal{P} \) and \( K \in \mathcal{K} \cap \mathcal{U} \).
Since \( g_m(z) \) is SOS and, hence, nonnegative, it follows that
\[
0 \leq g_m(z)
\]
\[
= f_m(z) - \xi(K) - \sum_{i=1}^{n_a} a_i(p)\alpha_{mi}(z) - \sum_{j=1}^{n_b} b_j(p)\beta_{mj}(z) - \sum_{l=1}^{n_c} c_l(z)\gamma_{ml}(z)
\]
\[
\leq f_m(z) - \xi(K)
\]
since $a_i(p) \geq 0$ and $b_j(p) = 0$ as $p \in \mathcal{P}$, $c_l(z) \geq 0$ as $K \in \mathcal{K} \cap \mathcal{U}$, and $\alpha_{mi}(z) \geq 0$ and $\gamma_{ml}(z) \geq 0$ as these polynomials are SOS. Hence,

$$\xi(K) \leq f_m(z) \quad \forall K \in \mathcal{K} \cap \mathcal{U} \forall p \in \mathcal{P} \forall m = 1, \ldots, n_f.$$ 

Therefore,

$$\xi(K) \leq \inf_{m=1,\ldots,n_f} f_i(z) \quad \forall K \in \mathcal{K} \cap \mathcal{U},$$

which implies (38). Lastly, suppose that $\xi^*(K) > 0$ for some $K \in \mathcal{K} \cap \mathcal{U}$. From (38) one has

$$s(K) > 0.$$ 

Hence, from (29) it follows that $K \in \mathcal{K}_s$, i.e., (39) holds. \hfill \Box

Theorem 2 states that a lower bound $\xi^*(K)$ of the function $s(K)$ in (28) over $\mathcal{K} \cap \mathcal{U}$ can be obtained by solving the SOS program (37), which is a convex optimization problem since the constraints are equivalent to LMI's, see [4] for details about SOS polynomials. If there exists $K$ in the set $\mathcal{K} \cap \mathcal{U}$ such that $\xi^*(K) > 0$, then $K$ is a robust stabilizing controller for the system (7), i.e., $K \in \mathcal{K}_s$. Section V will address the determination of such a controller.

Let us observe that the controller-dependent lower bound $\xi(K)$ is obtained in Theorem 2 by maximizing the integral of $\xi(K)$ over $\mathcal{K}$, i.e., $h(\xi)$. This is done in order to minimize the gap between $s(K)$ and $\xi(K)$, and exploits the idea we introduced in [3] for estimating the parameter-dependent lower bound of estimates of the robust domain of attraction.

Let us also observe that the degree of the polynomials $\xi(K)$, $\alpha_{mi}(z)$, $\beta_{mj}(z)$ and $\gamma_{ml}(z)$ can be freely chosen in Theorem 2. A possibility is to choose the degree of $\xi(K)$, i.e., $d_\xi$, and define the minimum degree of $\alpha_{mi}(z)$ as

$$d_{\alpha_{mi}} = \max \{ \deg(f_m), \deg(a_i), \deg(b_j), \deg(c_l) \}, \quad i = 1, \ldots, n_a, \quad j = 1, \ldots, n_b, \quad l = 1, \ldots, n_c \}. \quad (40)$$

Then, the degree of $\alpha_{mi}(z)$, $\beta_{mj}(z)$ and $\gamma_{ml}(z)$ are chosen as the largest degrees such that the degree of $g_m(z)$ is even and not greater than $d_{\alpha_{mi}}$. Clearly, the degree of $\alpha_{mi}(z)$ and $\gamma_{ml}(z)$ must be even since these polynomials are required to be SOS in Theorem 2. This choice will be adopted in the remaining part of the paper.
V. ROBUST STABILIZING CONTROLLER

The idea for establishing the non-emptiness of $K_s$ and solving (13) is to minimize $r(K)$ over the set of admissible controllers for which $\xi^*(K)$ is positive, i.e.,

$$\hat{r}^* = \inf_K r(K)$$

subject to

$$\xi^*(K) \geq \varepsilon$$
$$K \in K \cap U$$

where $\varepsilon > 0$ is introduced for considering positive values only of $\xi^*(K)$. Define the polynomial

$$w(z) = r(K) - \theta - (\xi^*(K) - \varepsilon)\delta(z) - \sum_{i=1}^{n_a} a_i(p)\alpha_i(z) - \sum_{j=1}^{n_b} b_j(p)\beta_j(z) - \sum_{l=1}^{n_c} c_l(z)\gamma_l(z)$$

where $\theta \in \mathbb{R}$ and $\alpha_i(z)$, $\beta_j(z)$, $\gamma_l(z)$ and $\delta(z)$ are polynomials. Define the SOS program

$$\theta^* = \sup_{\theta, \alpha_i, \beta_j, \gamma_l, \delta} \theta$$

subject to

$$w(z) \text{ is SOS}$$
$$\alpha_i(z) \text{ is SOS } \forall i = 1, \ldots, n_a$$
$$\gamma_l(z) \text{ is SOS } \forall l = 1, \ldots, n_c$$
$$\delta(z) \text{ is SOS}.$$  \hspace{1cm} (43)

Similarly to Theorem 2, (43) is a convex optimization problem, and

$$\theta^* \leq \hat{r}^*.$$  \hspace{1cm} (44)

The degree $\alpha_i(z)$, $\beta_j(z)$, $\gamma_l(z)$ and $\delta(z)$ can be freely chosen, and one can adopt a rule similar to the one provided for choosing the degree of the polynomials in Theorem 2. Indeed, let us define the minimum degree of $w(K)$ as

$$d_w = \max \{\deg(r), \deg(\xi), \deg(a_i), \deg(b_j), \deg(c_l), i = 1, \ldots, n_a, j = 1, \ldots, n_b, l = 1, \ldots, n_c\}.$$  \hspace{1cm} (45)

Then, the degree of $\alpha_i(z)$, $\beta_j(z)$, $\gamma_l(z)$ and $\delta(z)$ are chosen as the largest degree such that the degree of $w(z)$ is even and not greater than $d_w + \nu$, where $\nu$ is a free nonnegative integer. Clearly, the degree of $\alpha_i(z)$, $\gamma_l(z)$ and $\delta(z)$ must be even since these polynomials are required to be SOS in Theorem 2. The numerical examples in Section VII are solved with the choice $\nu = 0$.

**Theorem 3**: Let $w^*(z)$ be the optimal value of $w(z)$ in (43). One has

$$\theta^* = \hat{r}^*.$$  \hspace{1cm} (46)
if and only if there exist $K$ and $p$ such that

$$
\begin{align*}
\begin{cases}
w^*(z) & = 0 \\
r(K) & = \theta^* \\
\xi^*(K) & \geq \varepsilon \\
K & \in \mathcal{K} \cap \mathcal{U} \\
p & \in \mathcal{P}.
\end{cases}
\end{align*}
$$

(47)

In such a case, $K$ is a robust stabilizing controller for the system (7), i.e., $K \in \mathcal{K}_s$. Moreover, $\theta^*$ is an upper bound of the optimal cost $r^*$, i.e., $\theta^* \geq r^*$.

Proof. \(\Rightarrow\) Let us suppose that $\theta^* = \hat{r}^*$. Let $K^*$ be the maximizer of (41) and observe that

$$
\begin{align*}
\begin{cases}
r(K^*) & = \hat{r}^* \\
\xi^*(K^*) & \geq \varepsilon \\
K^* & \in \mathcal{K} \cap \mathcal{U}.
\end{cases}
\end{align*}
$$

Let us also observe that

$$
\begin{align*}
w^*(z) = r(K) - \theta^* - (\xi^*(K) - \varepsilon)\delta^*(z) - \sum_{i=1}^{n_a} a_i(p)\alpha_i^*(z) - \sum_{j=1}^{n_b} b_j(p)\beta_j^*(z) - \sum_{l=1}^{n_c} c_l(z)\gamma_l^*(z)
\end{align*}
$$

where $\alpha_i^*(z)$, $\beta_j^*(z)$, $\gamma_l^*(z)$ and $\delta^*(z)$ are the polynomials $\alpha_i(z)$, $\beta_j(z)$, $\gamma_l(z)$ and $\delta(z)$ found with (43). Let $z^*$ be $z$ defined for $K$ replaced by $K^*$ and for $p \in \mathcal{P}$. One has

$$
\begin{align*}
w^*(z^*) = -(\xi^*(K^*) - \varepsilon)\delta^*(z) - \sum_{i=1}^{n_a} a_i(p)\alpha_i^*(z^*) - \sum_{j=1}^{n_b} b_j(p)\beta_j^*(z^*) - \sum_{l=1}^{n_c} c_l(z^*)\gamma_l^*(z^*) \leq 0
\end{align*}
$$

since $\xi^*(K^*) - \varepsilon \geq 0$, $a_i(p) \geq 0$, $b_j(p) = 0$, $c_l(z^*) \geq 0$, $\alpha_i^*(z^*) \geq 0$, $\gamma_l^*(z^*) \geq 0$ and $\delta^*(z^*) \geq 0$. Let us observe that (43) ensures that $w^*(z)$ is a SOS polynomial. Hence,

$$
0 \leq w^*(z^*) \leq 0
$$

which implies that $w^*(z^*) = 0$. Therefore, (47) holds with $K$ replaced by $K^*$ for some $p \in \mathcal{P}$.

\(\Leftarrow\) Suppose that (47) holds for some $K$ and $p$. Since (44) implies that $\theta^*$ is a lower bound of $\hat{r}^*$, and (47) implies that this lower bound is achieved, one has $\theta^* = \hat{r}^*$.

Lastly, let us observe that, if there exist $K$ and $p$ such that (47) holds, one has $\xi^*(K) > 0$. Since $K \in \mathcal{K} \cap \mathcal{U}$, it follows from (39) that $K \in \mathcal{K}_s$. Consequently, $\theta^*$ is an upper bound of the optimal cost $r^*$ because there exists $K \in \mathcal{K}_s$ such that $r(K) = \theta^*$. \(\square\)
Theorem 3 suggests that one can obtain a robust stabilizing controller for the system (7) by looking for $K$ and $p$ such that (47) holds. If there exist such $K$ and $p$, then $\theta^*$ coincides with $\hat{r}^*$ and is an upper bound of $r^*$. Moreover, such a $K$ is a robust stabilizing controller for the system (7) if $\xi^*(K) > 0$.

In order to look for $K$ and $p$ such that the condition (47) holds, one can first compute the set
\[
C = \{ z : w^*(z) = 0 \}. \tag{48}
\]
The computation of $C$ can be addressed via linear algebra operations since $w^*(z)$ is SOS, in particular by looking for vectors of monomials in $z$ that belong to the null space of a positive semidefinite Gram matrix of $w^*(z)$, see [6], [14]. Once $C$ has been found, one checks whether any $z$ in the set $C$ satisfies the condition (47), hence extracting the controller $K$ from such $z$.

VI. CONVERGENCE

Theorem 4: Suppose that $\mathcal{P}$ is compact and $\mathcal{K}_s \neq \emptyset$. Suppose also that $a_i(p)$ and $b_i(p)$ have even degree, and that their highest degree homogeneous parts have no common zeros except 0. Then, there exist $\tau \in \mathcal{T}$ and sufficiently large integers $d_{\xi}$ (degree of the controller-dependent lower bound) and $\nu$ (defined under (45)) such that $\mathcal{N} = \emptyset$ and the condition (47) holds for some $K$ and $p$ by choosing $c_{l0}(z)$ such that $c_l(z)$ in (33) have even degree. Moreover, $\theta^*$ in (43) asymptotically converges to $r^*$ in (13).

Proof. Suppose that $\mathcal{K}_s \neq \emptyset$. Then, the robust stabilizability function $s(K)$ in (28) satisfies $s(K) > 0$ for some $K \in \mathcal{K} \cap \mathcal{K}_{wp}$. For any of such $K$, let us define $\tau$ as
\[
\tau = \begin{cases} 
0 & \text{if } \det(I - D(p) K) \geq \rho_{wp} \text{ for all } p \in \mathcal{P} \\
1 & \text{otherwise}.
\end{cases}
\]
It follows that $K \in \mathcal{K} \cap \mathcal{U}$. Let us define
\[
\bar{\xi}(K) = \arg \sup_{\xi \in \Xi} h(\xi)
\]
s.t. $\xi(K) \leq s(K)$ $\forall K \in \mathcal{K} \cap \mathcal{U}$
where $\Xi$ is the set of polynomials $\mathbb{R}^{m_u \times m_y} \to \mathbb{R}$ of degree not greater than $d_{\xi}$. Since $\mathcal{K} \cap \mathcal{U}$ is compact, it follows that $\bar{\xi}(K)$ approximates arbitrarily well $s(K)$ over $\mathcal{K} \cap \mathcal{U}$ for $d_{\xi}$ sufficiently large. This means that there exists $d_{\xi}$ even such that, for some $K \in \mathcal{K} \cap \mathcal{U}$,
\[
\bar{\xi}(K) > 0.
\]
Next, for any chosen degree of $\alpha_{mi}(z)$, $\beta_{mj}(z)$ and $\gamma_{ml}(z)$, the constraints in (37) provide a sufficient condition through the Positivstellensatz for establishing whether (38) holds. Since $P$ and $K \cap U$ are compact, and due to the structure of the polynomials $a_i(p)$, $b_i(p)$ and $c_i(z)$, this condition is also necessary for polynomials $\alpha_{mi}(z)$, $\beta_{mj}(z)$ and $\gamma_{ml}(z)$ of degree sufficiently large, see [12]. Since the choice of the degrees of these polynomials at the end of Section IV allows one to increase the degrees of all the polynomials in (37) by simply increasing the degree of $\xi(K)$, it follows that there exists $d_\xi$ even such that, for some $K \in K \cap U$,

$$\xi^*(K) > 0.$$ 

Similarly, for any nonnegative integer $\nu$ and for any degree of the polynomials $\alpha_i(z)$, $\beta_j(z)$, $\gamma_l(z)$ and $\delta(z)$, the constraints in (43) provide a sufficient condition through the Positivstellensatz for establishing whether (44) holds. As in the previous case, this condition is also necessary for polynomials $\alpha_i(z)$, $\beta_j(z)$, $\gamma_l(z)$ and $\delta(z)$ of degree sufficiently large. Since the choice of the degrees of these polynomials after (44) allows one to increase the degrees of all the polynomials in (43) by simply increasing $\nu$, it follows that there exists $\nu$ such that

$$\theta^* = \hat{r}^*.$$ 

From Theorem 3, this implies that there exist $K$ and $p$ such that (47) holds.

Lastly, in order to show that $\theta^*$ asymptotically converges to $r^*$, let $K^*$ be the maximizer in (13), and let us redefine $\tau$ as previously done in this proof with $K$ replaced by $K^*$. It follows that the controller-dependent lower bound $\xi^*(K)$ approximates arbitrarily well $s(K)$ over $K \cap U$ for $d_\xi$ sufficiently large as previously explained in this proof. Moreover,

$$\{K \in K \cap U : s(K) > 0\} = \mathcal{K}_a \cap U$$

i.e., the constraint in the optimization problem (41) coincides with a portion of $\mathcal{K}_a$ that contains the maximizer $K^*$ of (13). □

The controller-dependent lower bound found in (37) is obtained by maximizing $h(\xi)$, i.e., the integral of $\psi(K)\xi(K)$ over $\mathcal{K}$. Indeed, maximizing $h(\xi)$ has the effect of maximizing $\xi(K)$ and, hence, the set of admissible controllers for which $\xi^*(K)$ is positive. Since the goal is to minimize the cost $r(K)$ according to (13), one does not need to increase $\xi(K)$ everywhere over
\( \mathcal{K} \), but only (or especially) in the portion of \( \mathcal{K} \) where \( r(\mathcal{K}) \) is small. This suggests to choose \( \psi(\mathcal{K}) \) large where \( r(\mathcal{K}) \) is small, for instance according to

\[
\psi(\mathcal{K}) = \psi_0 - r(\mathcal{K})
\]

where \( \psi_0 \in \mathbb{R} \) is any constant such that \( \psi(\mathcal{K}) \) is positive over \( \mathcal{K} \). This choice is adopted in the example in the next section.

VII. Example

The computations in this example are done in Matlab on a standard computer (Windows 7, Intel Core 2, 3 GHz, 4 GB Ram) using the toolbox SeDuMi [16]. Consider (1)–(6) with

\[
A(p) = \begin{pmatrix}
0 & 3 & -1 - p_2^2 \\
2p_1 & -2 + p_2 & 0 \\
1 + 2p_2 & 0 & 1
\end{pmatrix}, \quad B(p) = \begin{pmatrix}
0 & 0 \\
2 + p_2 & -p_1^2 \\
0 & 2
\end{pmatrix}
\]

\[
C(p) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad D(p) = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}, \quad \mathcal{P} = \{ p \in \mathbb{R}^2 : p_1^2 + p_2^2 \leq 1 \}
\]

\( \rho_s = 0.1, \ k_{ij}^- = -5, \ k_{ij}^+ = 5, \ r(\mathcal{K}) = k_{11}^2 + k_{22}^2 - k_{12}^2 - k_{21}^2. \)

The autonomous system is not robustly stable since for \( p = (0, 0)' \) one has \( \text{spec}(A(p)) = \{-2, 0.5 \pm 0.866\sqrt{-1}\} \). Since \( D(p) \) is null it follows that

\[
\mathcal{U} = \begin{cases}
\mathbb{R}^{2 \times 2} & \text{if } \tau = 0 \\
\emptyset & \text{if } \tau = 1.
\end{cases}
\]

Indeed, the system is well-posed for all \( \mathcal{K} \). Hence, we consider only the case \( \tau = 0 \). It turns out that \( \mathcal{N} = \emptyset \). We solve (37) for \( d_\xi = 2 \) (the number of LMI scalar variables and the computational time are 692 and 3.4s). Hence, we solve (43) with the found \( \xi^*(\mathcal{K}) \), which provides \( \theta^* = 31.223 \) (the number of LMI scalar variables and the computational time are 21 and 0.4s). Next, we find that (47) holds with

\[
K = \begin{pmatrix}
-3.558 & -1.124 \\
-0.291 & -4.463
\end{pmatrix}.
\]

Therefore, \( K \) is a robust stabilizing controller, and \( \theta^* \) is an upper bound of the optimal cost \( r^* \).

This upper bound can be improved by increasing \( d_\xi \). Indeed, we solve (37) for \( d_\xi = 4 \) (the number of LMI scalar variables and the computational time are 1078 and 9.0s). Hence, we solve
(43) with the found $\xi^*(K)$, which provides $\theta^* = -4.611$ (the number of LMI scalar variables and the computational time are 232 and 0.7s). Next, we find that (47) holds with

$$K = \begin{pmatrix} -2.374 & -0.384 \\ 5.000 & -3.860 \end{pmatrix}.$$ 

VIII. Conclusion

This paper has addressed the design of robust static output feedback controllers for systems with parametric uncertainties. The proposed method is based on the introduction of the class of robust stabilizability functions, approximation of one of them through a polynomial lower bound, and derivation of a robust stabilizing controller candidate that minimizes a given cost. Due to the numerical complexity, at present, only systems with small dimension and few uncertain parameters can be considered. The proposed method is asymptotically nonconservative under mild assumptions, and can be implemented directly using solvers for SOS programming, or indirectly (working out the LMI conditions for establishing SOS polynomials) using solvers for semidefinite programming.

REFERENCES


