Frequently Asked Questions about Why Jackson's Theorem Is Incorrect

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We have disproved Jackson's theorem with two counterexamples [2]. The following are our answers to a collection of frequently asked questions, raised in discussions and reviews on our previous work related to this issue. For some questions, the relevant background is given before the answer.

Question 1: Why is the single-server queue with feedback not an ordinary M/M/1 queue without feedback?

Background: Consider Jackson's feedback queue, i.e., an M/M/1 queue with instantaneous Bernoulli feedback. The external arrival rate is λ. The service rate is μ. Before eventually leaving the system, any customer may return to the queue once served, with probability θ. Denote by $X(t)$ the number of customers waiting and being served in the queue at time t. As we have shown, this queue is actually a counterexample to Jackson's theorem; the flaw in Jackson's proof has been identified [2].

A remedy is proposed to salvage Jackson's theorem. This remedy argues that, for the M/M/1 queue with feedback, the equation, given by Jackson himself in his proof [3], is missing one term; the "correct" equation is

$$P\{X(t+h)=k\} = [1-\lambda h-\mu(1-\theta)h]P\{X(t)=k\} + \mu(1-\theta)hP\{X(t)=k+1\}$$
$$+\lambda hP\{X(t)=k-1\}+o(h).$$

Since the above equation is *identical* to the equation governing the evolution of the probability law of an ordinary M/M/1 queue without feedback, with arrival rate λ, and the service rate $\mu(1-\theta)$, why is the feedback queue not an ordinary M/M/1 queue without feedback?

Answer: The equation given in the proposed remedy alters completely Jackson's original feedback queue; it does not hold for Jackson's model. Jackson himself has never claimed that his feedback queue is an ordinary M/M/1 queue without feedback. The overall arrival process of the feedback queue is not Poisson. But by definition, the arrival process of an ordinary M/M/1 queue without feedback must be Poisson. Clearly, the two queues are not the same.

The ordinary M/M/1 queue without feedback may have a steady state. However, the existence of a steady state of the feedback queue is just *claimed* in Jackson's paper [3], based on the *assumption* that $X(t)$ is a stationary Markov process. As indicated by the flaw in Jackson's proof, $X(t)$ is not a Markov process, which has also been proved without diagnosing Jackson's proof [2]. Consequently, Jackson actually failed to prove the existence of a steady state for his feedback queue.

Therefore, in this special case of Jackson's feedback queue, any proposed remedy for Jackson's theorem must then begin with proving the existence of a steady state for *the* queue with feedback. Such a proof must strictly follow Jackson's definition of his model, and Jackson's feedback queue should not be replaced by any ordinary M/M/1 queue without feedback. In particular, $X(t)$ should not be assumed to be a Markov process.

In fact, we have proved that the feedback queue does not have a steady state. Please see the answer to Question 12.

Question 2: In the literature, there exist different proofs of Jackson's theorem. What are the flaws in the proofs?

Answer: All known "proofs," e.g., the "proof" with time reversibility, are based on the assumption that the system state is a stationary Markov process, or on other similar, incorrect assumption.

Question 3: The feedback queue is used in the literature to explain Jackson's theorem. What is wrong with the explanation?

Answer: The explanation is also based on the incorrect assumption that $X(t)$ is a stationary Markov process.

Question 4: "Although the feedback queue is not an ordinary M/M/1 queue without feedback, it behaves as an ordinary M/M/1 queue without feedback in steady state." Why is this assertion incorrect?

Answer: It is incorrect since the feedback queue does not have a steady state [2]. Please also see the answer to Question 12.

Question 5: "In steady state, a stable M/M/1 queue can be modeled as a time-reversible Markov process. According to time reversibility, the departure process of the queue is a Poisson process." What is wrong with the above argument?

Background: We have shown that the departure process of a stable M/M/1 queue in steady state has both a marginal, stationary version, and a non-marginal, non-stationary version. Only the non-marginal version can be directly observed [2]. The Poisson departure process is the marginal version, which is *constructed*, either analytically or experimentally, by averaging out the impact of the state of the queue on the output.

Consider two consecutive departure epochs, denoted by s and t, where $s < t$. At time s, the server is exclusively either busy or idle. It is intuitively clear that, if the server is busy, the inter-departure time $(t - s)$ is sampled from the service-time distribution; if the server is idle, the inter-departure time is distributed as the sum of an idle time of the server and a service time. To visualize this fact, we may color the inter-departure times sampled from the service-time distribution red, and color the inter-departure times distributed as the sum of the idle and service times blue. Any sample path of the inter-departure time sequence, which is the directly-observable, non-marginal version, is now represented by a sequence of colored time segments.

Answer: If we observe directly the inter-departure time sequence of the queue in steady state by looking backwards in time, we still see the colored time segments, just as what we see by looking forward in time. So what we can directly observe, by looking either backwards or forward in time, is the non-marginal version, which is not a Poisson process. Time reversibility might be used to show that the marginal version is a Poisson process; it should not be used to deny the existence of the non-marginal version.

Question 6: "Send me the colored inter-departure times, but don't let me see the color of any segment. Then I shall receive independent and identically distributed exponential inter-departure times without color." Is the above argument correct?

Answer: No. The color of any inter-departure time never disappears, regardless of whether one chooses to see it or not. The different colors indicate that the inter-departure times are sampled from different distributions. The observed tendency for the inter-departure times with the same color to aggregate shows that the inter-departure times are not mutually independent. Actually, the i.i.d.

exponential sequence is *constructed* by using the colored segments as raw materials [2].

Question 7: "The departure process of a stable M/M/1 queue in steady state has only one version, which is not Poisson with respect to the filtration of the arrival and service process, and is Poisson with respect to its own filtration." Is such argument correct?

Answer: No. The existence of the non-marginal version is simply a fact, as visualized by the sequences of the colored segments, which represent its sample paths. Clearly, with respect to whatever filtration, a sequence of the colored segments is not a sample path of a Poisson process.

Question 8: "A stable M/M/1 queue in steady state has only one departure process, whose rate (complete intensity) is not a constant with respect to the past of the whole queue, and is a constant with respect to the past of the departure process itself. It is the latter that shows that the departure process is Poisson." Is such argument correct?

Answer: No. The existence of the non-marginal version, with its sample paths visualized by the sequences of the colored segments, can be directly observed [2]. The non-marginal version is clearly not Poisson. Since it is impossible for the same process to be and not to be Poisson, the Poisson process must be a different process. In fact, the Poisson process is *constructed* based on the non-marginal version, by averaging out the impact of the state of the queue, and hence is referred to as the marginal version [2].

Question 9: "For a stable M/M/1 queue in steady state, the departure process is stationary. Why is there a non-stationary version?"

Answer: The non-stationary version is the non-marginal version. It is not stationary due to the impact of the state of the queue. Such impact persists forever, even in steady state. The existence of the non-marginal, non-stationary version does not necessarily exclude the marginal, stationary version. By averaging out of the impact of the state of the queue, the marginal version can be constructed. This marginal version is the stationary departure process.

Question 10: Given that there are two different versions of the departure process, what does Burke's theorem really mean?

Answer: Burke's theorem means that the marginal version is the Poisson departure process. Actually, Burke's theorem was motivated by isolating queues (referred to by Burke also as "stages") in a tandem system, rather than analyzing the queues jointly, as said by Burke himself [1]:

"It is intuitively clear that, in tandem queuing processes of the type mentioned above, if the output distribution of each stage was of such character that the queuing system formed by the second stage was amenable to analysis, then the tandem queue could be analyzed stage-by-stage insofar as the separate delay and queue-length distributions are concerned. Such a stage-by-stage analysis can be expected to be considerably simpler than the simultaneous analysis heretofore necessary. Fortunately, under the conditions stated below, it is true that the output has the required simplicity for treating each stage individually."

Such isolation is possible only based on the marginal version, and only the isolated queue is amenable to a *steady-state* analysis.

Question 11: What if two M/M/1 queues in tandem are considered jointly? In general, what if queues in a Jackson network are considered jointly?

Answer: If the two queues are considered jointly, the arrival process at the downstream queue is the non-marginal, non-stationary version of the departure process of the upstream queue. Consequently, the downstream queue is unstable, in the sense that the behavior of the queue cannot be described by a time-independent probability law. As a result, the tandem system as a whole is not stable. Due to the same reason, if the queues in the network are considered jointly, the network is not stable.

Question 12: Why is the queue with feedback unstable?

Answer: The occurrence of feedback depends intrinsically on the state of the queue, e.g., feedback cannot occur at time t, if $X(t) = 0$. In other words, the impact of the state of the queue on feedback cannot be averaged out. So, the feedback process is part of the non-marginal version of the departure process. Since the non-marginal version is not stationary, neither is the feedback process. The overall arrival process of the queue is the aggregation of the external Poisson arrival process and the feedback process. Since the feedback process is not stationary, neither is the aggregated arrival process. As a result, the behavior of the queue cannot be described by a time-independent probability law. In this sense, the feedback queue is not stable. The instability does not necessarily mean that $X(t)$ will increase towards infinity as $t \to \infty$. Analogous to a bounded sequence that does not converge to a limit, $X(t_1), X(t_2), \cdots$ for all $t_1 < t_2 < \cdots$ do not converge in distribution to a limit random variable, although $X(t)$ can be finite at any time t.

Question 13: By allowing any returning customer to enter service again immediately rather than waiting in line, Jackson's feedback queue can be modified without affecting the number of customers in the queue. Why is this modified queue not an ordinary M/M/1 queue without feedback?

Background: This question is essentially the same as Question 1. For the feedback queue, a customer may be served more than once. The *actual service time* required by a customer is then an initial service time, plus several, if there is any, extra service times due to feedback. In the original feedback queue, and in the modified queue, the actual service time of any customer remains unchanged. The puzzle is why the actual service time is not exponential with parameter $\mu(1-\theta)$ (see the background of Question 1).

Our answer to Question 1 is based on the analysis given in [2]. By answering this current question, we shall solve the puzzle by means of a (thought) experiment, similar to that suggested in [2]. Please review Question 1, its background, and our answer to that question. Please also see the background of Question 5.

Answer: For given λ, μ, and θ (see the background of Question 1), let us simulate Jackson's feedback queue, and observe the sequence of the actual service times of the customers. The simulation can simply be done as a thought experiment, and we encourage the reader to do the simulation. In particular, we encourage the reader to do it as a thought experiment now, so we can solve the puzzle together.

In the experiment, we count the number of service actually required by every customer. The actual service time of a customer is sampled, when the customer eventually departs and never returns. If the sampled actual service time consists of only one realized value of the exponential random variable with parameter μ, we color this actual service time red. If the sampled actual service time consists of at least two realized values of the exponential random variable with parameter μ, we color it blue. Now any sample path of the sequence of the actual service times is visualized by a sequence of the colored time segments.

Such sample paths are what we can see in the experiment. The different colors indicate that the actual service times do not follow any common distribution. A red actual service time is distributed differently from a blue actual service time. It is not even necessary for blue actual service times to follow any common distribution. A blue actual service time is distributed as the sum of a *finite* number of i.i.d. exponential random variables. Another blue actual service time can be distributed as a sum of a different, but also *finite* number of i.i.d. exponential random variables.

Clearly, the sequence of the actual service time sequence is not i.i.d. exponential. Therefore, it is incorrect to say that the sequence of the actual service times of the modified queue (or the original feedback queue) is i.i.d. exponential, since the sample paths of the actual service time sequence, which are used to simulate the queue, and are observed in the experiment, are not sample paths of any i.i.d. exponential sequence. Consequently, the modified queue and hence the feedback queue are not ordinary M/M/1 queues without feedback.

By modifying the original feedback queue, the feedback process is removed from the overall arrival process of the feedback queue. So the arrival process of the modified queue is the external Poisson arrival process of the feedback queue and hence is stationary. Nevertheless, the modified queue is still unstable (see Question 12 and our answer to that question), since its sequence of (actual) service times is not stationary, as shown by the different colors of the actual service times. As can be seen above, the cause of the instability is feedback. Unless feedback is prohibited, the queue cannot be stable. Indeed, if we do not allow any customer to be served more than once, the original queue and the modified queue become an identical queue, which is an ordinary M/M/1 queue without feedback, and can be stable.

Before we finish our thought experiment, we point out that an i.i.d. exponential sequence with parameter $(1-\theta)\mu$ may be *constructed*, either analytically or experimentally, by averaging out the impact of feedback on the actual service times. The experimental construction is similar to the construction of the marginal version of the departure process of a stable M/M/1 queue in steady state [2].

As shown by our thought experiment, this *constructed* sequence is neither relevant to the feedback queue nor to the modified queue: Any actual service time sampled in the experiment is not a realized value of an exponential random variable with parameter $(1-\theta)\mu$. A queue driven by Poisson arrivals with this *constructed* sequence as its service time sequence is indeed an ordinary M/M/1 queue without feedback. However, this M/M/1 queue should not be confused with Jackson's original feedback queue or with the modified queue.

Due to such confusion, any ordinary M/M/1 queue without feedback, whose service rate is μ', might be considered identical to a Jackson's feedback queue *at any time*, since the M/M/1 queue satisfies the equation given in the remedy for Jackson's theorem, discussed in the background of Question 1, by letting $\mu = \mu'/(1-\theta)$. This is not even claimed in Jackson's theorem [3]. Since θ in $\mu = \mu'/(1-\theta)$ can be arbitrary, the ordinary M/M/1 queue without feedback might even be considered identical to infinitely many different Jackson's feedback queue, an inexplicable result of misinterpreting Jackson's theorem.

Furthermore, the interpretation of Jackson's feedback queue as an ordinary M/M/1 queue without feedback also contradicts the explanation in the existing literature given by people considered experts in this field, although we do not agree with the existing explanation either, see Question 3 and our answer to that question.

References

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[3] J. R. Jackson, "Networks of waiting lines," *Operations Research*, vol. 5, no. 4, 518-521, 1957.