Noise in superresolution reconstruction

Edmund Y. Lam

Department of Electrical and Electronic Engineering, University of Hong Kong, Pokfulam Road, Hong Kong, China

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Recently there has been a significant interest in reconstructing a high-resolution (HR) image based on a set of low-resolution (LR) images with relative displacement. These images are typically undersampled with respect to the image spectrum of a HR image. I show that, although ideally a resolution increase of N times is possible with N LR images, in a practical system noise is a limiting factor that increases substantially as we approach this theoretical superresolution limit. For one dimension and a special case with two LR images, I present an analytical result of the noise amplification as a function of their displacement. This is defined as a condition number of the superresolution system, with the associated definitions of a well-conditioned and ill-conditioned superresolution reconstruction system.

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In the evaluation of a diffraction-limited imaging system, the resolution limit is an important metric for comparison. The Rayleigh criterion is the established practical measure for this purpose.[1] It is well known that the minimal resolvable separation δ for an incoherent imaging system is

$$\delta = 0.61(\lambda/\mathrm{NA}), \tag{1}$$

where λ is the wavelength and NA is the numerical aperture of the optical system. Attempts have been made to attain resolution beyond this diffraction limit, such as use of the bandwidth extrapolation proposed by Gerchberg and Papoulis.[2,3] However, these methods are known to be sensitive to noise in the measured data and are therefore of limited practical use.

More recently, an alternative approach to achieve image resolution beyond the Rayleigh limit has gained in popularity.[4–6] Instead of trying to achieve superresolution with one image, some researchers are focusing on the synthesis of multiple low-resolution (LR) images to form a high-resolution (HR) image, a procedure called superresolution reconstruction. The LR images cover the same scene but with subpixel shifts with respect to one another. As we will see, a resolution enhancement of N times is theoretically possible with N LR images, provided that no two images have identical overlap. However, as all practical systems have noise, if the shifts are too small noise is significantly amplified in the reconstructed HR image.

To simplify analysis and the presentation of results, I will treat the one-dimensional reconstruction problem in what follows. I also assume that image detection is done with an ideal photodetector performing a point sampling. For photodetectors with a finite size, we can consider image detection as first low-pass filtered with the shape of the photodetector before point sampling. This also limits the extent of superresolution, but a detailed analysis is beyond the scope of this paper.

Let the image intensity be represented by $g(x)$, sampled at regular intervals X apart, with one image capture. This is represented as a multiplication of $g(x)$ with a train-of-impulse comb function

$$\frac{1}{X}\,\mathrm{comb}\left(\frac{x}{X}\right) \triangleq \frac{1}{X}\sum_{n=-\infty}^{\infty}\delta\left(\frac{x}{X}-n\right) = \sum_{n=-\infty}^{\infty}\delta(x-nX), \tag{2}$$

where $\delta(\cdot)$ is the Dirac delta function. Assume that the maximum detectable frequency in $g(x)$ is f_x. According to the Whittaker–Shannon sampling theorem, $g(x)$ can be completely recovered from $1/X\,\mathrm{comb}(x/X)g(x)$ if $X < 1/2f_x$ by use of a sinc filter for interpolation. Mathematically, if $\hat{g}(x)$ denotes the reconstructed signal, we have[7]

$$\hat{g}(x) = \mathrm{sinc}\left(\frac{x}{X}\right) * \left[\frac{1}{X}\,\mathrm{comb}\left(\frac{x}{X}\right)g(x)\right]. \tag{3}$$

Now assume that we have two LR images with a shift of a between them. This is akin to interlaced sampling,[7] which is represented by the two functions $1/X\,\mathrm{comb}(x/X)$ and $1/X\,\mathrm{comb}[(x-a)/X]$. It is known that for complete recovery of $g(x)$ based on the samples, $X < 1/f_x$ would suffice.[7] Furthermore, let

$$A\left(\frac{x}{X}\right) \triangleq \mathrm{sinc}\left(\frac{2x}{X}\right) - (\pi\,\cot a\pi)\,\frac{x}{X}\,\mathrm{sinc}^2\left(\frac{x}{X}\right), \tag{4}$$

the interpolation can be achieved with[7]

$$\hat{g}(x) = A\left(\frac{x}{X}\right) * \left[\frac{1}{X}\,\mathrm{comb}\left(\frac{x}{X}\right)g(x)\right] + A\left(-\frac{x}{X}\right) * \left[\frac{1}{X}\,\mathrm{comb}\left(\frac{x-a}{X}\right)g(x)\right]. \tag{5}$$

The sampling and the interpolation filter are shown in Fig. 1. Note that Eq. (5) does not depend on the value

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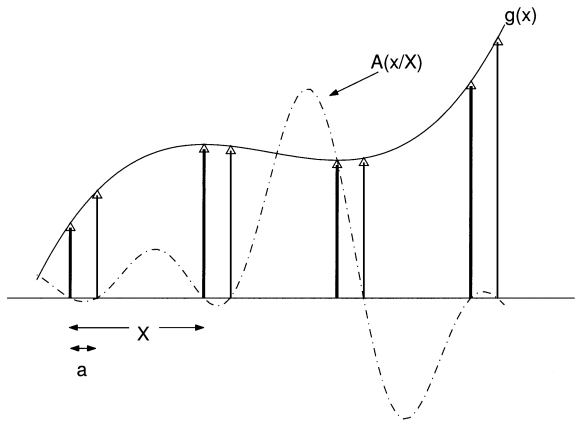

Fig. 1. Interlaced sampling.

of a as long as $0 < a < X$. Equation (5) can also be extended to N LR images by use of more complicated interpolation filters,[7,8] with which it is possible to resolve frequency as long as $f_x < (N/2X)$. Even in the limiting case when $a \to 0$, we arrive at a situation known as slope sampling, where we have the first $N - 1$ derivatives together with the sample values at each sample point.[9] When N approaches infinity, the conclusion can be somewhat startling. If we have absolute information about a point, including its value and all its derivatives, we essentially can reconstruct the whole image to any resolution. Of course, this is basically the Taylor series expansion

$$g(x) = g(x_0) + (x - x_0)g'(x_0) + \frac{(x - x_0)^2}{2!} g''(x_0) + \dots . \tag{6}$$

This idea has been exploited for superresolution with analytic continuation.[1] The Taylor series is known to be sensitive to noise in the samples, so if the displacements in the LR images are small, superresolution is also noise sensitive. Here the discussion is limited to only two LR images. Without loss of generality, we assume that $X = 1$. Furthermore, because of symmetry, we consider $0 < a < 0.5$ only.

At each sensor location, we sample $g(x) + n(x)$, where $n(x)$ is a noise term. To model electronic noise, where the dominant source is the random thermal motion of electrons, we can usually use an additive white Gaussian noise model, i.e., $n(x) \sim \mathcal{N}(0, \sigma)$.[10] In the range between 0 and 1 we can compute the noise amount after interpolation by using Eq. (5). Let ${\sigma_{a,b}}^2$ be the noise content at interpolated location b. In that case,

$${\sigma_{a,b}}^2 = \sum_{n=-\infty}^{\infty} [A^2(b + n) + A^2(a - b + n)]\sigma^2. \tag{7}$$

To determine how this equation behaves with different values of a, we plot the situations with $a = 0.3$ and $a = 0.15$ in Fig. 2. Each plot shows the noise amount at various locations between 0 and 1. Since we need two samples spaced 0.5 apart in the HR image within this range, the average noise content is

$${\sigma_a}^2 = \frac{1}{2}\left({\sigma_{a,b}}^2 + {\sigma_{a,b+0.5}}^2\right). \tag{8}$$

As can be seen in Figs. 2(a) and 2(b), a sample taken between 0 and a would have a smaller noise content than the samples, due to averaging of the noise. However, the other reconstruction location must then be between a and 1, where the noise content is magnified. Their average, which increases when a decreases, is a constant regardless of their locations.

This average can be computed analytically with some algebraic manipulation. Note that it suffices to compute for $b = 0$ and that ${\sigma_{a,0}}^2 = \sigma^2$. Let

$$Q_1 \triangleq \sum_{n=-\infty}^{\infty} A^2(0.5 + n), \tag{9}$$

$$Q_2 \triangleq \sum_{n=-\infty}^{\infty} A^2(a - 0.5 + n), \tag{10}$$

then ${\sigma_a}^2 = {}^1\!/_2(1 + Q_1 + Q_2)\sigma^2$. For Q_1 an equivalent

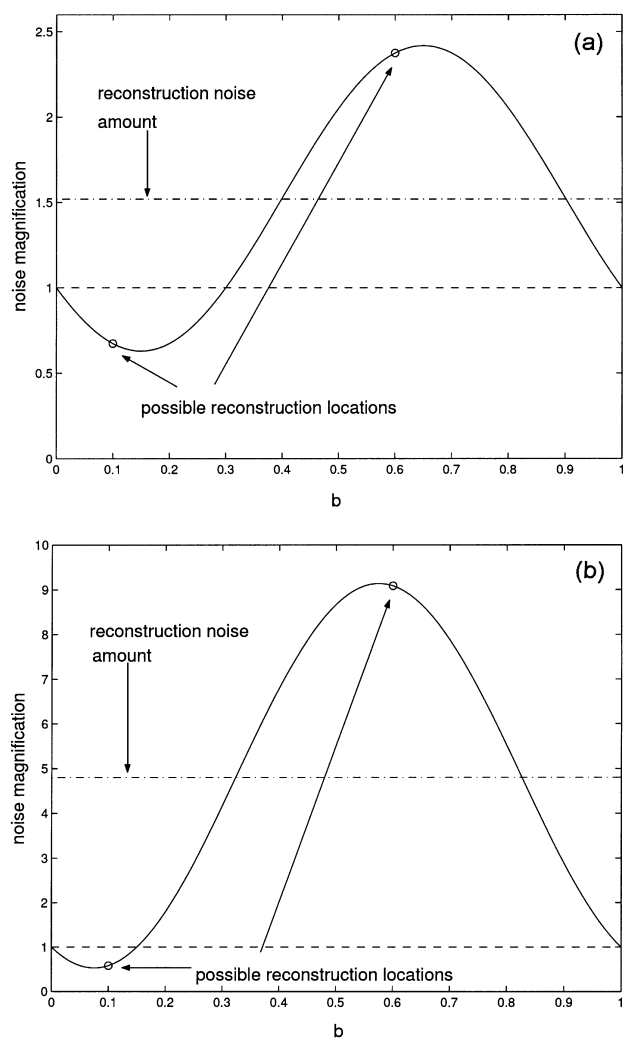

Fig. 2. Noise magnification in interlaced sampling. Reconstruction with (a) $a = 0.3$ and (b) $a = 0.15$.

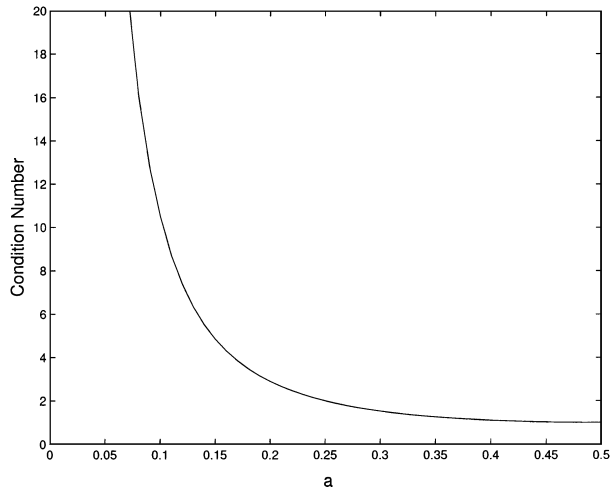

Fig. 3. Condition number of the superresolution reconstruction system.

expression is

$$Q_1 = \int_{-\infty}^{\infty} A^2(x)\mathrm{comb}(x - 0.5)\mathrm{d}x$$
$$= [\tilde{A}(s) * \tilde{A}(s) * \exp(-j\pi s)\mathrm{comb}(s)]_{s=0}\,, \qquad (11)$$

where we have made use of the Fourier transform property that $\int_{-\infty}^{\infty} A(x)\mathrm{d}x = \tilde{A}(0)$ with $A(x)$ and $\tilde{A}(s)$ as a Fourier transform pair. $\tilde{A}(s)$ is computed to be[7]

$$\tilde{A}(s) = \begin{cases} -\dfrac{m}{1-m} = c & 0 < s < 1 \\ -\dfrac{\overline{m}}{1-\overline{m}} = d & -1 < s < 0\,, \\ 0 & \text{otherwise} \end{cases} \qquad (12)$$

where $m = \exp(-j2\pi a)$ and $\overline{m} = \exp(j2\pi a)$. We define c and d to simplify the mathematics presentation. Therefore, $\tilde{A}(s) * \tilde{A}(s)$ has a value of c^2 at $s = 1$, $2cd$ at $s = 0$, and d^2 at $s = -1$. The value of Q_1 is therefore

$$Q_1 = c^2 \exp(j\pi) + 2cd + d^2 \exp(-j\pi)$$
$$= -c^2 + 2cd - d^2. \qquad (13)$$

Similarly, we can compute Q_2 to be

$$Q_2 = -c^2\overline{m} + 2cd - d^2 m\,. \qquad (14)$$

Noise content is therefore

$$\sigma_a{}^2 = \frac{1}{2}[1 - (1 + \overline{m})c^2 + 4cd - (1 + m)d^2]\sigma^2$$
$$= \frac{1}{2}\left[1 - \frac{m^2 + 6m + 1}{(1-m)^2}\right]\sigma^2$$
$$= \frac{4\sigma^2}{(1-m)(1-\overline{m})}\,. \qquad (15)$$

We therefore define the quantity

$$K \triangleq \frac{4}{[1 - \exp(-j2\pi a)][1 - \exp(j2\pi a)]} \qquad (16)$$

as the condition number of the superresolution reconstruction system. As $a \to 0.5$, $K \to 1$ and the system is well-conditioned with no noise amplification. In this case, the two images are exactly half a pixel apart. As $a \to 0$, $K \to \infty$ and the system is ill-conditioned. A small amount of noise in the LR image would be significantly amplified in the reconstructed HR image. The value of K with various values of a is plotted in Fig. 3.

A superresolution reconstruction system in which multiple LR images are used to form a HR image has been investigated. It has been shown that, in the presence of noise, the relative displacement of the LR images has major implications on noise amplification. For a simple system with only two LR images, I presented a simple analytical formula that indicates whether the system is well-conditioned or ill-conditioned. This formula will serve as an aid as more complex systems with multiple LR images at various displacements are investigated.

E. Lam's e-mail address is elam@eee.hku.hk.

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