Estimating the Domain of Attraction for Non-Polynomial Systems via LMI optimizations

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Abstract

This paper proposes a strategy for estimating the DA (domain of attraction) for non-polynomial systems via LFs (Lyapunov functions). The idea consists of converting the non-polynomial optimization arising for a chosen LF in a polynomial one, which can be solved via LMI optimizations. This is achieved by constructing an uncertain polynomial linearly affected by parameters constrained in a polytope which allows us to take into account the worst-case remainders in truncated Taylor expansions. Moreover, a condition is provided for ensuring asymptotical convergence to the largest estimate achievable with the chosen LF, and another condition is provided for establishing whether such an estimate has been found. The proposed strategy can readily be exploited with variable LFs in order to search for optimal estimates. Lastly, it is worth to remark that no other method is available to estimate the DA for non-polynomial systems via LMIs.

1 Introduction

It is well-known that estimating the DA (domain of attraction) of an equilibrium point is a problem of fundamental importance in systems engineering. It is also well-known that the DA is a complicated set which does not admit an analytic representation in most cases, see for example [2–5]. Therefore, looking for inner approximations of the DA via estimates with simple shape has become a fundamental issue since long time. A common way of obtaining such estimates is based on Lyapunov stability theory. Specifically, given a LF (Lyapunov function) proving local asymptotical stability of the equilibrium, any sublevel set of this function included in the region where
its temporal derivative takes negative values is guaranteed to be an inner estimate of the DA.

In recent years, various and efficient have been developed for estimating the DA for polynomial systems, mainly based on LMI relaxations for solving polynomial optimizations [6–10]. These methods are interesting because they amount to solving a convex optimization with LMI constraints for a chosen LF, and because their conservatism can be arbitrarily decreased by increasing the degree of the relaxation. See also [11] and papers therein.

Unfortunately, most real systems are non-polynomial systems, consider for instance pendulums, chemical reactors, and systems with saturations. Some methods have been proposed to establish whether an equilibrium point of a non-polynomial system is stable, such as [12] which propose some changes of coordinate, and [13] which proposes the use of polynomial approximations. However, no method has been proposed to estimate the DA for non-polynomial systems via LMIs, which is still an open problem.

This paper proposes a strategy for estimating the DA via LMI optimizations. The idea consists of converting the non-polynomial optimization arising for a chosen LF in a polynomial one, which can be solved through convex optimizations based on LMI relaxations. This polynomial optimization is obtained by expressing the non-polynomial terms via truncated Taylor expansions and parameterizing their remainders inside a convex polytope. This allows us to take into account the worst-case remainders by simply considering only the vertices of the polytope. Moreover, the conservatism of the proposed strategy is investigated, in particular by proposing a condition which ensures asymptotical convergence to the largest estimate achievable with the chosen LF, and another condition which allows one to establish whether such an estimate has been found. The proposed strategy can be readily exploited with variable LFs in order to search for optimal estimates.

The paper is organized as follows. Section 2 introduces the problem formulation. Section 3 describes the proposed strategy. Section 4 presents some illustrative examples. Lastly, Section 5 reports some final remarks.

2 Preliminaries

2.1 Problem formulation

Notation: \( \mathbb{N}, \mathbb{R} \): natural and real numbers sets; \( 0_n \): origin of \( \mathbb{R}^n \); \( I_n \): identity matrix \( n \times n \); \( A' \): transpose of matrix \( A \); \( A > 0 \) (\( A \geq 0 \)): symmetric positive definite (semidefinite) matrix \( A \); \( \text{tr}(A) \): trace of matrix \( A \); \( \text{ver}(\mathcal{P}) \): set of vertices of the polytope \( \mathcal{P} \); s.t.: subject to.
Let us consider the continuous-time nonlinear system

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + \sum_{i=1}^{r} g_i(x(t))\xi_i(x_{a_i}(t)), \\
x(0) &= x_{\text{init}}
\end{align*}
\]

where \(x(t) = (x_1(t), \ldots, x_n(t))^\prime \in \mathbb{R}^n\) is the state, \(x_{\text{init}} \in \mathbb{R}^n\) is the initial condition, the functions \(f, g_1, \ldots, g_r : \mathbb{R}^n \to \mathbb{R}^n\) are polynomial, \(a_1, \ldots, a_r \in \{1, \ldots, n\}\) are indexes, and the functions \(\xi_i : \mathbb{R} \to \mathbb{R}\) are non-polynomial.

It is assumed that the origin is the equilibrium point of interest. The DA of the origin is the set of initial conditions for which the state asymptotically converges to the origin, and it is indicated by

\[
\mathcal{D} = \left\{ x_{\text{init}} \in \mathbb{R}^n : \lim_{t \to +\infty} x(t) = 0 \right\}.
\]

In the sequel the dependence on the time \(t\) will be omitted for ease of notation.

Let \(v : \mathbb{R}^n \to \mathbb{R}\) be a continuously differentiable, positive definite and radially unbounded function, and suppose that \(v\) is a LF for the origin in (1), i.e. the time derivative of \(v\) along the trajectories of (1) is locally negative definite. Then, the sublevel set

\[
\mathcal{V}(c) = \{ x \in \mathbb{R}^n : v(x) \leq c \} \setminus \{0_n\}
\]

is an estimate of \(\mathcal{D}\) if

\[
\dot{v}(x) < 0 \quad \forall x \in \mathcal{V}(c).
\]

Let us assume that \(v\) is polynomial of degree \(2m\). The problems addressed in this paper are as follows:

1. for a chosen LF, computing the largest estimate \(\mathcal{V}(c^*)\) where

\[
c^* = \sup \{ c \in \mathbb{R} : (4) \text{ holds} \} ;
\]

2. for a variable LF, searching for optimal estimates according to a chosen criterion, i.e. solving

\[
\varrho^*_{2m} = \sup_{v \in \mathcal{F}_{2m}} \zeta(\mathcal{V}(c^*))
\]

where \(\mathcal{F}_{2m}\) is the set of polynomials of degree \(2m\) (in \(n\) variables), and \(\zeta : \mathbb{R}^n \to \mathbb{R}\) is a measure of \(\mathcal{V}(c^*)\) representing the chosen criterion.

In the sequel we will assume that the first \(\delta\) derivatives of \(\xi_i(x_{a_i})\) are continuous on

\[
\mathcal{V}_{a_i}(c) = \{ x_{a_i} \in \mathbb{R} : x \in \mathcal{V}(c) \}.
\]

**Remark 1.** It should be remarked that solving (5) is an unavoidable step for solving (6).
2.2 Polynomial optimization via LMIs

Before proceeding let us briefly describe how LMIs can be used for solving polynomial optimizations. Let \( x^{(m)} \) be a vector containing all monomials of degree less than or equal to \( m \) in \( x \). Then, \( v \) can be written as

\[
v(x) = x^{(m)'} (V + L(\alpha)) x^{(m)}
\]

where \( V \) is any symmetric matrix such that \( v = x^{(m)'} V x^{(m)} \), \( L \) is any linear parametrization of the set \( \mathcal{L} = \{ L = L' : x^{(m)'} L x^{(m)} = 0 \} \), and \( \alpha \) is a free vector. This representation is known as Gram matrix \([14]\) and SMR (square matricial representation) \([6]\). The expression (8) was introduced in \([6]\) in order to investigate positivity of polynomials via LMIs: indeed, \( v \) is SOS (sum of squares of polynomials) if and only if

\[
\exists \alpha : V + L(\alpha) \geq 0
\]

which is an LMI feasibility test. This can be used in the search for estimates of the domain of attraction in the case of polynomial systems and LFs. Indeed, in such a case the sublevel set \( \mathcal{V}(c) \) satisfies (4) if there exists a polynomial \( s \) such that

\[
s \text{ and } -\dot{v} + (v - c)s
\]

are SOS and vanish only for \( x = 0_n \)

which amounts to solving an LMI feasibility test. The degree of \( s \) is initially selected small, typically equal to the minimum even value which does not increase the degree of the polynomial \(-\dot{v} + (v - c)s \). If this choice does not allow to compute \( c^* \) (such a condition can be detected for instance as done in \([15, 16]\)), then one increases this degree and repeat the computation.

Analogously, (4) can be investigated via LMIs through moments relaxations \([17]\) which are dual to SOS relaxations. See also the Matlab toolboxes GloptiPoly and SOSTOOLS where these relaxations are implemented.

3 Estimates computation

3.1 Fixed Lyapunov function

Let \( v \) be a LF satisfying the assumption in Section 2.1. Let \( k, k \leq \delta \), be a positive integer, and let us rewrite \( \xi_i \) via Taylor expansion of degree \( k \) as

\[
\xi_i(x_{a_i}) = h_i(x_{a_i}) + w_i \frac{x_{a_i}^k}{k!}
\]
where \( w_i \in \mathbb{R} \) is a parameter to be selected and

\[
h_i(x_{a_i}) = \sum_{j=0}^{k-1} \frac{d^j \xi_i(x_{a_i})}{dx_{a_i}^j} \bigg|_{x_{a_i}=0} x_{a_i}^j.
\] (12)

Let us introduce the polynomials

\[
p(x) = \frac{\partial v(x)}{\partial x} \left( f(x) + \sum_{i=1}^{r} g_i(x) h_i(x_{a_i}) \right)
\] (13)

\[
q_i(x) = \frac{\partial v(x)}{\partial x} g_i(x) \frac{x_{a_i}^k}{k!}
\] (14)

\[
q(x) = (q_1(x), \ldots, q_r(x))'.
\] (15)

**Theorem 1** Let \( c_k \) be the solution of the polynomial optimization

\[
c_k = \sup_{c \in \mathbb{R}} c \quad \text{s.t.} \quad p(x) + q(x)'w < 0 \quad \forall x \in \mathcal{V}(c) \quad \forall w \in \text{ver}(\mathcal{W})
\] (16)

where \( \mathcal{W} \) is the rectangle

\[
\mathcal{W} = [\sigma_1, -, \sigma_1, +] \times \ldots \times [\sigma_r, -, \sigma_r, +]
\] (17)

and \( \sigma_i, -, \sigma_i, + \in \mathbb{R} \) are any bounds satisfying

\[
\sigma_i, - \leq \frac{d^k \xi_i(x_{a_i})}{dx_{a_i}^k} \leq \sigma_i, + \quad \forall x_{a_i} \in \mathcal{V}_{a_i}(c).
\] (18)

Then, \( c_k \leq c^* \).

**Proof** Suppose that the constraint in (16) is fulfilled, i.e.

\[
p(x) + q(x)'w < 0 \quad \forall x \in \mathcal{V}(c) \quad \forall w \in \text{ver}(\mathcal{W}).
\] (19)

Then, let us observe that \( p \) is the Taylor expansion of \( \dot{v} \) truncated at degree \( k - 1 \), while \( q'w \) is the remainder of this truncation in the Lagrange form where \( w \) is a suitable vector in the rectangle \( \mathcal{W} \). Indeed, from the property of this remainder one has that

\[
\forall x \in \mathcal{V}(c) \\exists w_i \in [\sigma_i, -, \sigma_i, +] : \xi_i(x_{a_i}) = h_i(x_{a_i}) + w_i x_{a_i}^k
\] (20)

and, hence,

\[
\forall x \in \mathcal{V}(c) \\exists w \in \mathcal{W} : \dot{v}(x) = p(x) + q(x)'w.
\] (21)
Therefore, one has from (19)–(21) that

\[
\dot{v}(x) < 0 \quad \forall x \in \mathcal{V}(c)
\]

because \( p + q'w \) is affine linear in \( w \) and \( \mathcal{W} \) is a convex polytope, i.e. the extremes of \( p + q'w \) for \( w \in \mathcal{W} \) are taken at the vertices of \( \mathcal{W} \).

Theorem 1 provides a strategy for computing a guaranteed lower bound \( c_k \) of the sought \( c^* \) by solving a polynomial optimization which can be handled by LMIs as explained in Section 2.2. This strategy consists of expressing \( \xi_i(x_{a_i}) \) via Taylor expansion of degree \( k \) and parameterizing their remainders inside a convex polytope. This allows us to take into account the worst-case remainders by simply considering only the vertices of the polytope. As a result, for any \( k \) the set \( \mathcal{V}(c_k) \) is guaranteed to be an inner estimate of \( \mathcal{D} \).

Another interesting and useful aspect is that the constraint in (16) is obtained without introducing any additional multiplier: indeed, the worst-case remainder is taken into account through a set of polynomial inequalities which are simply obtained by evaluating \( p + q'w \) at the vertices of the rectangle \( \mathcal{W} \). As explained in (17), these vertices are simply given by the bounds \( \sigma_{i,-}, \sigma_{i,+} \) fulfilling (18).

Let us observe that the computation of \( \sigma_{i,-}, \sigma_{i,+} \) requires the analysis of one-variable function over an interval, which is typically a non-difficult task. Moreover, the set \( \mathcal{V}_{a_i}(c) \) can be readily overestimated via eigenvalues analysis by representing \( v \) as in (8). See for instance the examples in Section 4.

The solution of (16) can be found in several ways. One consists of a bisection on \( c \) where at each step the bounds \( \sigma_{i,-}, \sigma_{i,+} \) are computed based on \( c \) and a corresponding LMI optimization is solved. Another way consists of defining the bounds \( \sigma_{i,-}, \sigma_{i,+} \) for an over-estimate of \( c_k \), and hence solving one only LMI optimization. The former method is slower but less conservative than the latter one, and it is adopted for the examples in Section 4.

Lastly, let us observe that the complexity can be an issue for large number of variables and degree of the polynomials. This is probably the “cost to pay” for achieving estimates of the domain of attraction via convex programming. Nevertheless, the computational time required for small-scale system is indeed reasonable as shown by the examples in Section 4.
3.2 Conditions for non-conservatism

A natural question hence arises: does the lower bound \( c_k \) asymptotically converge to the sought \( c^* \)? The following result provides an answer to this question.

**Theorem 2** Assume \( \delta = \infty \) and \( c^* < \infty \), and suppose there exists \( \tau \in \mathbb{R} \), \( \tau < \infty \), such that the bounds \( \sigma_{i-}, \sigma_{i+} \) in Theorem 1 satisfy also

\[
\begin{cases}
|\sigma_{i-}| < \tau \\
|\sigma_{i+}| < \tau
\end{cases}
\quad \forall i = 1, \ldots, r \quad \forall c \leq c^* \quad \forall k.
\]

Then,
\[
\lim_{k \to \infty} c_k = c^*.
\]

**Proof** Let us observe that \( c_k \) is given by (16) while \( c^* \) is given by (5). These optimizations have the same cost function, instead the constraint for the former is
\[
p(x) + q(x)'w < 0 \quad \forall x \in V(c) \quad \forall w \in \text{ver}(W)
\]
while the constraint for the latter is
\[
\dot{v}(x) < 0 \quad \forall x \in V(c).
\]

Now, from (21) one has that, for all \( x \in V(c) \), there exists \( \bar{w}(x) \in \mathcal{W} \) such that
\[
p(x) + q(x)'w - \dot{v}(x) = q(x)'w - q(x)'\bar{w}(x).
\]
Hence, taking into account (23), one has that, for all \( x \in V(c) \),
\[
|p(x) + q(x)'w - \dot{v}(x)| \leq 2\tau \sqrt{r} \sup_{x \in V(c)} \|q(x)\|.
\]

Lastly, let us observe that from (13)–(15) one has
\[
q(x) = \frac{\bar{q}(x)}{k!}
\]
where \( \bar{q} \) is a vector of polynomials, which implies that for all \( x \in V(c) \) with any finite \( c \) one has that
\[
\lim_{k \to \infty} \left|p(x) + q(x)'w - \dot{v}(x)\right|
\leq \lim_{k \to \infty} 2\tau \sqrt{r} \sup_{x \in V(c)} \|q(x)\|
\leq 2\tau \sqrt{r} \lim_{k \to \infty} \frac{1}{k!} \sup_{x \in V(c)} \|\bar{q}(x)\|
= 0
\]
because $\|\bar{q}\|$ is finite over any finite domain such as $\mathcal{V}(c)$. Therefore, (16) and (5) coincide as $k$ goes to infinity, and hence the theorem holds. □

Theorem 2 provides a condition for asymptotic non-conservatism of the lower bound $c_k$, which consists of defining the rectangle $\mathcal{W}$ through quantities $\sigma_{i,-}, \sigma_{i,+}$ bounded in absolute value by a finite constant for all $c \leq c^*$. Let us observe that, in some cases, the existence of such a constant $\tau$ is easily guaranteed, for instance for trigonometric functions $\xi_i$ (such as sine and cosine) and for saturation functions. Moreover, this existence can be ensured also for unbounded functions such as $\xi_i(x_n) = e^{x_n}$: indeed, one has that the $k$th derivative is $e^{x_n}$ and the existence of $\tau$ is automatically implied since $c^*$ is assumed finite.

Lastly, it would be clearly useful if one could establish whether the found $c_k$ is tight because the search for tighter values of $c_k$ could be immediately terminated. The following result provides a condition to answer to this question.

**Theorem 3** Let $\bar{x}$ be an optimal value of $x$ in (16), i.e. such that

\[
\begin{align*}
&v(\bar{x}) = c_k \\
&p(\bar{x}) + q(\bar{x})'w = 0 \quad \text{for some } w \in \text{ver}(\mathcal{W}).
\end{align*}
\]

Suppose that

\[\dot{v}(\bar{x}) = 0.\]

Then, $c_k = c^*$.

**Proof** Suppose (32) holds and suppose for contradiction that $c_k < c^*$. Consider any $c$ in $(c_k, c^*]$. One has that $\bar{x} \in \mathcal{V}(c)$ because $v(\bar{x}) = c_k < c$ and that $\dot{v}(\bar{x}) = 0$, hence implying that the sublevel set $\mathcal{V}(c)$ cannot contain only points $x \neq 0_n$ where $\dot{v}(x)$ is negative. Therefore, $c^*$ cannot be the optimal solution of (5), which leads to a contradiction. □

Theorem 3 provides a condition for ensuring that the found $c_k$ is the sought $c^*$. This condition simply requires to check whether any of the optimal values of $x$ in (16) annihilates the time derivative of $v(x)$. These optimal values are found from the eigenspaces of the Gram matrices returned by the LMI solver, see for instance [16] for details.

Lastly, let us remark that the proposed strategy can be extended to systems containing non-polynomial terms that are functions of more than one variable, i.e. having $g_i(x)$ instead of $g_i(x_n)$. The complications arising
in such a case concern the computation of the bounds $\sigma_{i,-}, \sigma_{i,+}$ fulfilling (18) since in this case the remainder will be a function of more than one variable. However, let us observe that these bounds do not need to be tight as explained in Theorems 1 and 2.

### 3.3 Variable Lyapunov function

In this section we address the solution of (6). First of all, it should be observed that the constraint (10) becomes a BMI (bilinear matrix inequality) if $v$ is variable (even in the simpler case of polynomial systems, see e.g. [7, 18]), and consequently the feasible set of the resulting optimization is nonconvex. This means that, at present, it is possible to guarantee to find only suboptimal solutions for (6).

Let us start by considering the criterion $\zeta$ in (6). A typical choice for this criterion is

$$\zeta(\mathcal{V}(c^*)) = \text{volume of } \mathcal{V}(c^*)$$

which defines the search for the LF in $\mathcal{F}_{2m}$ providing the estimate with the largest volume. However, the volume of $\mathcal{V}(c^*)$ cannot be expressed via an explicit formula for non-quadratic LFs, moreover this criterion leads to an optimization where not only the feasible set but also the cost function is nonconvex also in the case of quadratic LFs, see e.g. [18] for the case of polynomial systems. In order to cope with this problem, it has been proposed to approximate the maximization of the volume of $\mathcal{V}(c^*)$ via

$$\min_{c^*} \frac{\text{tr}(V)}{c^*}$$

where $V$ is a matrix representing $v$ as in (8): in fact, for the case of quadratic LFs the volume is maximized by minimizing $\det(V)/(c^*)^n$, which somehow minimizes the eigenvalues of $V$, and this motivates the choice in (34) which yields a convex (in particular, linear) cost function.

Another typical choice for $\zeta$ is the size of a set of fixed shape (such as a sphere) included in $\mathcal{V}(c^*)$, i.e.

$$\zeta(\mathcal{V}(c^*)) = \max \left\{ z \in \mathbb{R} : v(x) \leq c^* \forall x : x'x \leq z^2 \right\}$$

see e.g. [7] for the case of polynomial systems. This choice allows one to increase $\mathcal{V}(c^*)$ according to some chosen geometric rule, and also to formulate the cost of the resulting optimization via a linear function. Indeed, (35) can be considered via

$$\max \ z \ \text{ s.t. } \ (c^* - v) + (x'x - z^2)s \text{ are SOS}$$

9
where $s$ is an auxiliary polynomial: in fact, any $z$ satisfying the above constraint is guaranteed to be a lower bound of $\zeta(V(c^*))$. Let us observe that the above constraint is a BMI due to the product between $z$ and $s$, however since $z$ is a scalar it can be addressed via quasi-convex optimizations as shown in [16].

For choices such as (33) and (35), one can readily obtain suboptimal solutions for the problem (6) by using a gradient-based search, where at each iteration the value $c^*$ is found as described in Sections 3.1 and 3.2. Indeed, one can simply parameterize $v$ according to

$$v(x) = \bar{v}'x^{(2m)}$$

where $\bar{v}$ is a vector containing the coefficients of $v$ with respect to $x^{(2m)}$ (which is a vector containing all monomials of degree less than $2m$ but those of degree 0 and 1 since $v$ has to be positive definite), and then maximize the function

$$b(\bar{v}) = \begin{cases} \zeta(V(c^*)) & \text{if } v \text{ is SOS} \\ 0 & \text{otherwise} \end{cases}$$

which allows one to get rid of non-positive LFs. The number of free parameters in $\bar{v}$ is given by the size of $x^{(2m)}$ minus 1 since $v$ is defined up to a positive scale factor, and is equal to

$$d = \frac{(n + 2m)!}{n!(2m)!} - n - 2.$$  

See also the examples in Section 4. Alternatively, one may solve a sequence of LMI optimizations by iterating each time between the LF and the auxiliary polynomials.

### 4 Examples

#### 4.1 Example 1

Let us consider the system

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 + 0.5(e^{2x_1} - 1) \\ \dot{x}_2 = -x_1 - x_2 + x_1x_2 + x_1 \cos x_1 \end{cases}$$

First, we consider the LF $v(x) = x_1^2 + x_2^2$. This system can be written in the form of (1) with

$$f(x) = \begin{pmatrix} -x_1 + x_2 \\ -x_1 - x_2 + x_1x_2 \end{pmatrix}$$

$$g_1(x) = 0.5, \quad \xi_1(x_{a_1}) = e^{x_{a_1}} - 1, \quad a_1 = 1$$

$$g_2(x) = x_1, \quad \xi_2(x_{a_2}) = \cos x_{a_2}, \quad a_2 = 1.$$
Let us consider for simplicity $k = 1$. We have:

\[
\frac{d\xi_1(x_{a1})}{dx_{a1}} = e^{x_{a1}}, \quad \frac{d\xi_2(x_{a2})}{dx_{a2}} = -\sin x_{a2}
\]

\[V_{a1}(c) = V_{a2}(c) = [-\sqrt{c}, \sqrt{c}].\]

Constants $\sigma_{i,-}, \sigma_{i,+}$ satisfying (18) can be chosen as

\[
\sigma_{1,-} = e^{-\sqrt{c}}, \quad \sigma_{1,+} = e^{\sqrt{c}} \\
\sigma_{2,-} = -z, \quad \sigma_{2,+} = z, \quad z = \begin{cases} \sin \sqrt{c} & \text{if } \sqrt{c} \leq \pi/2 \\ 1 & \text{otherwise} \end{cases}
\]

The same investigation can be repeated for $k > 1$, also observe from Theorem 2 that $c_k \to c^*$ for finite $c^*$. Table 1 shows some lower bounds $c_k$ obtained from Theorem 1 and the respective computational times (Matlab implementation with a standard personal computer, the degree of $p$ in (10) is 2). We have that the condition of Theorem 3 holds for $k = 6$, i.e. $c_6 = c^*$. This is also verified by Figure 1 which shows that the estimate $V(c_6)$ is tangent to the curve $\dot{v} = 0$.

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Table 1: Example 1: lower bounds $c_k$ and computational time $t_c$.

Figure 2 shows the results obtained with variable LFs of degree $2m = 2$ and $2m = 4$. These results are obtained with $\zeta$ as in (35) via gradient-based search where at each iteration the value $c^*$ is found as just described. The LF is parameterized as in (37), and the number of free parameters in $\bar{v}$ given by (39) is $d = 2$ ($2m = 2$) and $d = 11$ ($2m = 4$). The estimates are obtained by maximizing the function in (38) using as initialization the LF in Figure 1. We have obtained the lower bounds $\varrho_2 = 1.02$ and $\varrho_4 = 1.13$ of $\varrho_2^*$ and $\varrho_4^*$ respectively. It is interesting to observe that the suboptimal estimate obtained for $2m = 4$ is quite close to the boundary of the true DA as proved by the presence of a diverging trajectory (dashed line) initialized at a point located nearby the boundary of this estimate ("□" mark).
Figure 1: Example 1: boundary of $\mathcal{V}(c_6)$ (solid line) and curve $\dot{v}(x) = 0$ ("o" marks).

Figure 2: Example 1: boundaries of the suboptimal estimates for $2m = 2$ (inner solid line, corresponding to $\varrho_2$) and $2m = 4$ (outer solid line, corresponding to $\varrho_4$), boundary of the estimate in Figure 1 (dashdot line), and a diverging trajectory (dashed line) initialized at the "□" mark.
4.2 Example 2

Let us consider the whirling pendulum in Figure 3 described by

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k_f}{m_b} x_2 + \frac{\omega^2}{l_p} \sin x_1 \cos x_1 - \frac{g}{l_p} \sin x_1
\end{align*}
\]

where \(x_1\) is the angle with the vertical, \(k_f\) is the friction, \(m_b\) is the mass of the rigid arm of length \(l_p\) rotating at the angular velocity \(\omega\), and \(g\) is the gravity acceleration. We consider the numerical values \(k_f = 0.2\), \(m_b = 1\), \(\omega = 0.9\), \(g = 10\) and \(l_p = 10\).

We first consider the LF \(v(x) = x_1^2 + x_1 x_2 + 4x_2^2\). Table 2 shows the lower bounds \(c_k\) provided by Theorem 1 for some values of \(k\). From Theorem 3 we find that \(c_7\) is tight, i.e. \(c^* = 0.6990\), which is also proved by Figure 4.

<table>
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<td>(c_k)</td>
<td>0.5357</td>
<td>0.4144</td>
<td>0.5928</td>
<td>0.6711</td>
<td>0.6990</td>
</tr>
<tr>
<td>(t_c) [s]</td>
<td>3.8</td>
<td>3.6</td>
<td>5.0</td>
<td>5.5</td>
<td>8.5</td>
</tr>
</tbody>
</table>

Table 2: Example 2: lower bounds \(c_k\) and computational time \(t_c\).
Figure 4: Example 2: boundary of $\mathcal{V}(c_7)$ (solid line) and curve $\dot{v}(x) = 0$ ("o" marks).

Then, we consider the case of variable LF as done in Example 1 by using as initialization the LF in Figure 4, hence obtaining the lower bounds $\varrho_2 = 0.53$ and $\varrho_4 = 1.06$ and the corresponding estimates shown in Figure 5.

5 Conclusion

We have proposed a strategy for estimating the DA of equilibrium points for non-polynomial systems via LFs and LMIs, which is based on the use of Taylor expansions and the parametrization of their remainders inside a convex polytope. Moreover, we have proposed conditions for non-conservatism in the case of a fixed LF, which are important in order to obtain less conservative estimates in the case of a variable LF.

References

Figure 5: Example 2: boundaries of the suboptimal estimates for $2m = 2$ (inner solid line, corresponding to $\rho_2$) and $2m = 4$ (outer solid line, corresponding to $\rho_4$). The figure shows also the boundary of a the estimate in Figure 4 (dashdot line).


