Homogeneous polynomial Lyapunov functions
for robust stability analysis of LFR systems

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Abstract

In this paper the use of Homogeneous Polynomial Lyapunov Functions (HPLFs) for robust stability analysis of linear systems subject to time-varying parametric uncertainty, affecting rationally the state transition matrix, is investigated. Sufficient conditions based on Linear Matrix Inequalities (LMI) feasibility tests are derived for the existence of HPLFs, which ensure robust stability when the uncertain parameter vector is restricted to lie in a convex polytope. It is shown that HPLFs lead to results which are less conservative than those obtainable via Quadratic Lyapunov Functions (QLFs).
1 Introduction

Robust analysis of linear systems subject to time-varying parametric uncertainty is a fundamental issue since long time. Polytopic systems, i.e. systems where the uncertain parameter vector affects affinely the state transition matrix, are quite often employed to model uncertainty in order to exploit their convexity properties for simplifying the analysis. Several approaches for assessing robust stability of polytopic systems have been proposed, for example by using Quadratic Lyapunov Functions (QLFs) (see, e.g., [1]), piecewise quadratic Lyapunov functions (see, e.g., [2] and [3]), polyhedral Lyapunov functions (see, e.g., [4] and [5]), and, more recently, Homogeneous Polynomial Lyapunov Functions (HPLFs) (see, e.g., [6] and [7]).

For systems where the parameter vector does not affect affinely the state transition matrix, computationally efficient approaches for the general case have not been devised yet. Indeed, a typical approach is to embed these systems in polytopic ones in order to obtain computationally tractable, though conservative, stability results. One important exception concerns the case when the uncertain parameter vector affects rationally the state transition matrix. In this case, exploiting Linear Fractional Representations (LFRs) of these systems, some approaches have been proposed providing sufficient conditions based on QLFs (see [8] and references therein).

In this paper the use of HPLFs for robust analysis of systems where the uncertain parameter vector affects rationally the state transition matrix is investigated. Exploiting LFRs and the Complete Square Matricial Representation (CSMR) of homogeneous forms (see, e.g., [9] and [7]), sufficient conditions are derived in terms of Linear Matrix Inequality (LMI) feasibility tests for the existence of HPLFs of degree $2m$ ensuring robust stability, when the uncertain parameter is restricted to lie in a convex polytope. It is proven that less conservative results are obtained if $m > 1$ with respect to the case $m = 1$, i.e. the QLFs case. Also, conditions for computing lower bounds of robust stability margins are derived in terms of Generalized EigenValue Problems (GEVPs). Some numerical examples are presented for illustrative purposes.

Notation

$I_n$: identity matrix $n \times n$;
$0_{n \times m}$: $n \times m$ zero matrix;
$A^\top$: transpose of matrix $A$;
$A > 0$ ($A \geq 0$): symmetric positive definite (semidefinite) matrix $A$;
$\text{he}(A)$: $A + A^\top$;
A ⊗ B: Kronecker’s product of matrices A and B;
A^[m]: A ⊗ A ⊗ ... ⊗ A (m-th Kronecker’s power of A);
co\{x₁, ..., x_m\}: convex hull of vectors xᵢ defined as

\[ \text{co}\{x₁, ..., x_m\} = \left\{ x : x = \sum_{i=1}^{m} \lambda_i x_i, \sum_{i=1}^{m} \lambda_i \leq 1, \lambda_i \geq 0 \right\}. \]

2 Problem formulation and preliminaries

Consider the uncertain linear time-varying system described by

\[ \dot{x}(t) = A_{rat}(\theta(t))x(t), \]  

where \( x(t) \in \mathbb{R}^n, A_{rat}(\theta(t)) \in \mathbb{R}^{n \times n} \) is a real-valued rational function of the time-varying parameter vector \( \theta(t) = (\theta₁(t), ..., \theta_r(t))' \in \mathbb{R}^r \). The parameter \( \theta(t) \) is assumed, for all \( t \geq 0 \), to lie in a polytope \( \Theta \subset \mathbb{R}^r \) containing the origin, i.e.,

\[ \theta(t) \in \Theta \triangleq \text{co}\{\theta^1, ..., \theta^l\}, \]

where \( \theta^i \in \mathbb{R}^r, i = 1, ..., l, \) are given vectors.

Throughout the paper, it is implicitly assumed that the function \( \theta(t) \) is such that the differential system (1) has a solution. Also, the explicit dependence on \( t \) will be omitted whenever it is convenient for easiness of notation.

The basic problem addressed in this paper is the construction of a Homogeneous Polynomial Lyapunov Function (HPLF), ensuring global asymptotic stability of system (1)-(2). More specifically, the aim is to find a homogeneous polynomial form of degree 2\( m \), denoted by \( v_{2m}(x) \), such that

(i) \( v_{2m}(x) > 0 \) for all \( x \neq 0 \);

(ii) \( \dot{v}_{2m}(x) < 0 \) for all \( x \neq 0 \) and for all \( \theta(t) \in \Theta \).

A possible way to tackle this problem is to compute a polytope \( \mathcal{A} \) bounding the set \( \{A_{rat}(\theta) : \theta \in \Theta\} \) and to employ the techniques for constructing HPLFs recently developed in [7]. However, since \( A_{rat}(\theta(t)) \) depends rationally on \( \theta \), this approach can in general lead to quite conservative results.

In this paper we pursue a different approach, based on Linear Fractional Representations (LFRs),
which has been used in [8] for the case of Quadratic Lyapunov Functions (QLFs), i.e., \( m = 1 \). More specifically, exploiting LFRs, system (1) can be equivalently rewritten as follows:

\[
\begin{cases}
\dot{x} &= Ax + Bq \\
p &= Cx + Dq \\
q &= E(\theta)p
\end{cases}
\]  

(3)

where \( A \in \mathbb{R}^{n \times n} \) is a Hurwitz matrix, \( E(\theta) \in \mathbb{R}^{d \times d} \) has the block-diagonal structure

\[
E(\theta) = \text{diag}(\theta_1 I_{s_1}, \ldots, \theta_r I_{s_r}),
\]  

(4)

\( q, p \in \mathbb{R}^d \) are auxiliary vectors, \( B \in \mathbb{R}^{n \times d}, C \in \mathbb{R}^{d \times n}, D \in \mathbb{R}^{d \times d} \) are suitable matrices, and \( d = s_1 + \ldots + s_r \). The LFR degree of system (3)-(4) is defined by the quantity

\[
d_{LFR} = \max\{s_1, \ldots, s_r\}.
\]  

(5)

We denote by \( \mathcal{E} \) the polytope of matrices

\[
\mathcal{E} = \left\{ E(\theta) \in \mathbb{R}^{d \times d} : \theta \in \Theta \right\}
\]  

(6)

and by \( E_i = E(\theta^i), i = 1, \ldots, l \), the vertices of \( \mathcal{E} \).

The problems we consider can be now stated as follows.

- **P1**: determine a HPLF \( v_{2m}(x) \) which ensures global asymptotic stability of (3)-(4) for all \( \theta(t) \in \Theta \); if such a HPLF exists, (3)-(4) is said to be \( 2m \)-homogeneously stable over \( \Theta \).

- **P2**: compute the robust stability margin

\[
\gamma_{2m} = \sup \{ \gamma : (3)-(4) \text{ is } 2m\text{-homogeneously stable over } \eta\Theta \text{ for all } \eta \in [0, \gamma] \}.
\]  

(7)

To proceed, we first recall the Complete Square Matricial Representation (CSMR) of homogeneous forms, which provides all the possible representations of a homogeneous form of degree \( 2m \) in terms of a quadratic form in the space of the monomials of \( x \) of degree \( m \) (see, for example, [9] and [7]). Let \( w_{2m}(x) \) be a homogeneous form of degree \( 2m \). The CSMR of \( w_{2m}(x) \) is defined as

\[
w_{2m}(x) = x^m (W_m + L_m(\alpha))x^m,
\]  

(8)
where \( x^{(m)} \in \mathbb{R}^{\sigma(n,m)} \) is a base vector of the homogeneous forms of degree \( m \) in \( x \) (containing all monomials of degree \( m \) in \( x \)), \( W_m \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)} \) is a suitable symmetric matrix, \( \alpha \in \mathbb{R}^{\sigma_{par}(n,m)} \) is a free vector, and \( L_m(\alpha) \) is a linear parameterization of the set
\[
L_m = \left\{ L_m = L'_m : \ x^{(m)}'L_m x^{(m)} = 0 \ \forall x \in \mathbb{R}^n \right\}.
\] (9)

The quantities \( \sigma(n,m) \) and \( \sigma_{par}(n,m) \) are given by
\[
\sigma(n,m) = \frac{(n + m - 1)!}{(n - 1)!m!},
\] (10)
\[
\sigma_{par}(n,m) = \frac{1}{2} \sigma(n,m)(\sigma(n,m) + 1) - \sigma(n,2m).
\] (11)

### 3 Robust stability analysis of LFR systems via HPLFs

In this section we provide sufficient conditions for problems P1 and P2 in terms of LMIs. In order to illustrate the main idea, let us first observe that from (3) one has
\[
\dot{x}^{(m)} = \frac{\partial x^{(m)}}{\partial x} \dot{x} = \frac{\partial x^{(m)}}{\partial x} (Ax + BE(\theta)p).
\] (12)

Hence, it is easy to see that \( \dot{x}^{(m)} \) is a vector whose elements are linear combination of the monomials contained in \( x^{(m)} \) and in \( p \otimes x^{(m-1)} \), i.e., there exist suitable matrices \( \bar{A}, \bar{B} \) such that
\[
\dot{x}^{(m)} = \bar{A} x^{(m)} + \bar{B} (p \otimes x^{(m-1)}).
\]

Now, consider the homogeneous form \( v_{2m}(x) = x^{(m)}'P_m x^{(m)} \) with \( P_m = P_m' > 0 \). It turns out that
\[
\dot{v}_{2m}(x) = x^{(m)}'(\bar{A}'P_m + P_m \bar{A})x^{(m)} + x^{(m)}'P_m \bar{B}(p \otimes x^{(m-1)}) + (p \otimes x^{(m-1)})'\bar{B}'P_m x^{(m)}
\]
and therefore \( \dot{v}_{2m}(x) \) can be written as a quadratic form in the vector
\[
y_m(x;p) \triangleq \begin{bmatrix} x^{(m)} \\ p \otimes x^{(m-1)} \end{bmatrix} \in \mathbb{R}^{\tau(n,d,m)}
\] (13)

where \( \tau(n,d,m) = \sigma(n,m) + d\sigma(n,m-1) \).

Vector \( y_m(x;p) \) plays the same role of the base vector \( x^{(m)} \) in the case of systems with polytopic uncertainties (see [7] for details). The introduction of the extended vector \( y_m(x;p) \) allows one to write down \( \dot{v}_{2m}(x) \) in a square matricial form like in (8), and to introduce a corresponding parameterization similar to (9). To this purpose, let us consider the set
\[
N_m = \left\{ N_m = N'_m : \ y'_m(x;p)N_m y_m(x;p) = 0 \ \forall x \in \mathbb{R}^n \ \forall p \in \mathbb{R}^d \right\}.
\] (14)
Lemma 1 The set $\mathcal{N}_m$ is a linear space of dimension
\[
\tau_{\text{par}}(n, d, m) = \frac{1}{2}\tau(n, d, m)(\tau(n, d, m)+1)-\left(\sigma(n, 2m) + d\sigma(n, 2m - 1) + \frac{1}{2}d(d + 1)\sigma(n, 2m - 2)\right).
\] (15)

Proof See Appendix A.1.

According to Lemma 1, $\mathcal{N}_m$ admits a linear parameterization $\mathcal{N}_m(\beta)$ where $\beta \in \mathbb{R}^{\tau_{\text{par}}(n, d, m)}$: a base of $\mathcal{N}_m$ can be computed as explained in Appendix A.2. As it will be shown in the following, this parameterization plays a key role in deriving sufficient conditions for problems P1 and P2, because it will allow us to express the condition $\hat{v}_{2m}(x) < 0$ in terms of LMIs, and hence to extend the results in [7] to systems in LFR form (3)-(4).

In particular, note that $\tau_{\text{par}}(n, d, 1) = 0$, i.e. $\mathcal{N}_m(\beta)$ is identically zero in the case of QLFs ($m = 1$).

Table 1 shows $\tau(n, d, m)$ and $\tau_{\text{par}}(n, d, m)$ for some other values of $m$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$n \backslash d$ & 1 & 2 & 3 & 4 \\
\hline
1 & 2; 0 & 3; 0 & 4; 0 & 5; 0 \\
\hline
2 & 5; 3 & 7; 6 & 9; 10 & 11; 15 \\
\hline
3 & 9; 14 & 12; 25 & 15; 39 & 18; 56 \\
\hline
4 & 14; 40 & 18; 66 & 22; 98 & 26; 136 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$n \backslash d$ & 1 & 2 & 3 & 4 \\
\hline
1 & 2; 0 & 3; 0 & 4; 0 & 5; 0 \\
\hline
2 & 7; 10 & 10; 21 & 13; 36 & 16; 55 \\
\hline
3 & 16; 72 & 22; 138 & 28; 225 & 34; 333 \\
\hline
4 & 30; 290 & 40; 519 & 50; 813 & 60; 1172 \\
\hline
\end{tabular}
\end{table}

Table 1: Quantities $\tau(n, d, m)$ and $\tau_{\text{par}}(n, d, m)$ for some values of $n, d, m$. Each cell has the form $\tau(n, d, m); \tau_{\text{par}}(n, d, m)$.

To proceed, we need to introduce the following matrices
\begin{align*}
\hat{A}_m(A) & \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)} : \frac{\partial x^{[m]}}{\partial x}Ax = \hat{A}_m(A)x^{[m]} \\
\hat{B}_m(B) & \in \mathbb{R}^{\sigma(n,m)\times d\sigma(n,m-1)} : \frac{\partial x^{[m]}}{\partial x}Bp = \hat{B}_m(B)\left(p \otimes x^{[m-1]}\right) \\
\hat{C}_m(C) & \in \mathbb{R}^{d\sigma(n,m-1)\times\sigma(n,m)} : Cx \otimes x^{[m-1]} = \hat{C}_m(C)x^{[m]} \\
\hat{D}_m(D) & \in \mathbb{R}^{d\sigma(n,m-1)\times d\sigma(n,m-1)} : Dp \otimes x^{[m-1]} = \hat{D}_m(D)\left(p \otimes x^{[m-1]}\right)
\end{align*}

where all the equalities hold $\forall x \in \mathbb{R}^n, \forall p \in \mathbb{R}^d$. Next lemma gives analytic expressions for the above matrices.
Lemma 2 Matrices $\bar{A}_m(A), \bar{B}_m(B), \bar{C}_m(C), \bar{D}_m(D)$ are given by

$$
\bar{A}_m(A) = (K'_m K_m)^{-1} K'_m \left( \sum_{j=0}^{m-1} I_{n_j} \otimes A \otimes I_{n^{m-1-j}} \right) K_m, \tag{20}
$$

$$
\bar{B}_m(B) = (K'_m K_m)^{-1} K'_m \left( \sum_{j=0}^{m-1} (I_{n_j} \otimes B) F_j \otimes I_{n^{m-1-j}} \right) (I_d \otimes K_{m-1}), \tag{21}
$$

$$
\bar{C}_m(C) = \left( C \otimes (K'_{m-1} K_{m-1})^{-1} K'_{m-1} \right) K_m, \tag{22}
$$

$$
\bar{D}_m(D) = D \otimes I_{\sigma(n,m-1)} \tag{23}
$$

where $K_m \in \mathbb{R}^{n^m \times \sigma(n,m)}$ is the matrix satisfying

$$
x[m] = K_m x[m] \quad \forall x \in \mathbb{R}^n \tag{24}
$$

and $F_j \in \mathbb{R}^{dn_j \times dn'}$ is given by

$$
F_j = \left[ \begin{array}{c}
I_d \otimes I_{n_j}(1,:) \\
\vdots \\
I_d \otimes I_{n_j}(n'_j,:) 
\end{array} \right], \tag{25}
$$

$I_{n_j}(i,:) \text{ being the } i\text{-th row of } I_{n_j}.$

Proof See Appendix A.3.

3.1 Sufficient conditions for problem P1

The next theorem provides a sufficient condition for the solution of problem P1, in terms of an infinite number of LMI.

Theorem 1 System (3)-(4) is $2m$-homogeneously stable over $\Theta$ if there exists $P_m \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$, $P_m = P^T_m > 0$, such that for all $E \in \mathcal{E}$ there exist $G_{m,E} \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m-1)}$, $H_{m,E} \in \mathbb{R}^{\sigma(n,m-1) \times \sigma(n,m-1)}$ and $\beta_E \in \mathbb{R}^{\gamma_{par}(n,d,m)}$ satisfying

$$
R_m(P_m, G_{m,E}, H_{m,E}, E) + N_m(\beta_E) < 0 \tag{26}
$$

where

$$
R_m(P_m, G_{m,E}, H_{m,E}, E) = \left[ \begin{array}{c}
\text{he}(P_m \bar{A}_m(A) + G_{m,E} \bar{C}_m(C)) \\
\bar{B}_m(B) P_m + \bar{D}_m(D) G_{m,E}' - G_{m,E}' + H_{m,E} \bar{C}_m(C) \\
P_m \bar{B}_m(B) + G_{m,E} \bar{D}_m(D) - G_{m,E} + \bar{C}_m(C) H_{m,E}' \\
\text{he}(H_{m,E} \bar{D}_m(D) - H_{m,E})
\end{array} \right], \tag{27}
$$

Proof See Appendix B.4.
Let us consider $v_{2m}(x) = x^{(m)}' P_m x^{(m)}$ for $P_m = P_m' > 0$. Then, from (3), (12) and (16)-(17) one has
\[
\frac{d}{dt} v_{2m}(x(t)) = y_m(x;p) \begin{bmatrix} \text{he}(P_m A_m(A)) & P_m B_m(BE) \\ B_m'(BE)P_m & 0 \end{bmatrix} y_m(x;p).
\]
From (3) and (18)-(19) it follows that
\[
p \otimes x^{(m-1)} = (Cx + DEp) \otimes x^{(m-1)} = \tilde{C}_m(C)x^{(m)} + \tilde{D}_m(DE)(p \otimes x^{(m-1)})
\]
and hence for any $G_{m,E}$ and $H_{m,E}$ one can write
\[
x^{(m)}' G_{m,E}(\tilde{C}_m(C)x^{(m)}) + \tilde{D}_m(DE)(p \otimes x^{(m-1)}) = x^{(m)}' G_{m,E}(p \otimes x^{(m-1)}) ,
\]
\[
(p \otimes x^{(m-1)})' H_{m,E}(\tilde{C}_m(C)x^{(m)}) + \tilde{D}_m(DE)(p \otimes x^{(m-1)}) = (p \otimes x^{(m-1)})' H_{m,E}(p \otimes x^{(m-1)}) .
\]
Summing the two equations above and the corresponding transpose, one gets
\[
y_m'(x;p)Xy_m(x;p) = 0
\]
where
\[
X = \begin{bmatrix} \text{he}(G_{m,E} \tilde{C}_m(C)) & G_{m,E} \tilde{D}_m(DE) - G_{m,E} + \tilde{C}_m(C)H_{m,E}' \\ \tilde{D}_m'(DE)G_{m,E}' - G_{m,E}' + H_{m,E} \tilde{C}_m(C) & \text{he}(H_{m,E} \tilde{D}_m(DE) - H_{m,E}) \end{bmatrix}.
\]
Then, recalling that $y_m'(x;p)N_m(\beta_E)y_m(x;p) = 0$ for any $\beta_E$ we have
\[
\frac{d}{dt} v_{2m}(x(t)) = y_m'(x;p) \begin{bmatrix} \text{he}(P_m A_m(A)) & P_m B_m(BE) \\ B_m'(BE)P_m & 0 \end{bmatrix} + X + N_m(\beta_E) y_m(x;p).
\]

Theorem 1 requires an infinite number of LMIs because the matrices $G_{m,E}$ and $H_{m,E}$ are allowed to change for different $E \in \mathcal{E}$. Sufficient conditions for $2m$-homogeneous stability based on a finite number of LMIs can be derived via a suitable choice of $G_{m,E}$ and $H_{m,E}$. The case of constant matrices is considered next.

**Corollary 1** System (3)-(4) is $2m$-homogeneously stable over $\Theta$ if there exist $P_m \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m)}$, $P_m = P_m' > 0$, $G_m \in \mathbb{R}^{\sigma(n,m)\times\sigma(n,m-1)}$, $H_m \in \mathbb{R}^{\sigma(n,m-1)\times\sigma(n,m-1)}$ and $\beta_i \in \mathbb{R}^{\tau_{m,n}(n,d,m)}$, $i = 1, \ldots, l$, such that
\[
R_m(P_m, G_m, H_m, E_i) + N_m(\beta_i) < 0, \quad i = 1, \ldots, l.
\] (28)
Proof Since the left-hand side of (26), for $G_{m,E} \equiv G_m$ and $H_{m,E} \equiv H_m$, depends affinely on $E$ and on the free parameter $\beta_E$, it is sufficient that the inequality is verified in the vertices $E_i$ of $E$. Hence, (28) follows.

If $d_{LFR} = 1$, a less stringent condition can be obtained as stated next.

**Corollary 2** Suppose that $d_{LFR} = 1$. Then, system (3)-(4) is $2m$-homogeneously stable over $\Theta$ if there exist $P_m \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$, $P_m' > 0$, $G_{m,i} \in \mathbb{R}^{\sigma(n,m) \times d\sigma(n,m-1)}$, $H_{m,i} \in \mathbb{R}^{d\sigma(n,m-1) \times d\sigma(n,m-1)}$ and $\beta_i \in \mathbb{R}^{\tau_{par}(n,d,m)}$, $i = 1, \ldots, l$, such that

$$R_m(P_m, G_{m,i}, H_{m,i}, E_i) + N_m(\beta_i) < 0, \quad i = 1, \ldots, l.$$  

(29)

Proof Inequality (29) implies that there exists a homogeneous form of degree $2m$, $v_{2m}(x) = x^{(m)}P_{m,x}^{(m)}$, with $P_m > 0$ such that, for $i = 1, \ldots, l$,

$$\left. \frac{d}{dt} v_{2m}(x)(t) \right|_{E=E_i} = x^{(m)}(t)Q(E_i)x^{(m)}(t)$$

where

$$Q(E_i) = \text{he}\left( P_m \bar{A}_m \left( A + BE_i(I - DE_i)^{-1}C \right) \right) + L_m(\alpha_i) < 0$$

and $\alpha_i$ depends on $\beta_i$. Since $d_{LFR} = 1$, it turns out (see [1]) that $\text{co}\{ A + BE_i(I - DE_i)^{-1}C, \quad i = 1, \ldots, l \} = \{ A + BE(I - DE)^{-1}C, \quad E \in \text{co}\{ E_1, \ldots, E_l \} \}$. Hence, since $\bar{A}_m(\cdot)$ is a linear function, it follows that $Q(E) < 0$ for any $E \in \text{co}\{ E_1, \ldots, E_l \}$, that is (3)-(4) is $2m$-homogeneously stable over $\Theta$.

The LMI feasibility tests (28) and (29) have dimension $\tau(n,m,d) \times \tau(n,m,d)$ and involve respectively $\zeta_1(n,d,m,l)$ and $\zeta_2(n,d,m,l)$ free parameters, where

$$\zeta_1(n,d,m,l) = \frac{1}{2} \sigma(n,m)(\sigma(n,m) + 1) + d\sigma(n,m)\sigma(n,m - 1) + d^2\sigma^2(n,m - 1) + l\tau_{par}(n,d,m) - 1$$

$$\zeta_2(n,d,m,l) = \frac{1}{2} \sigma(n,m)(\sigma(n,m) + 1) + d\sigma(n,m)\sigma(n,m - 1) + d^2\sigma^2(n,m - 1) + \tau_{par}(n,d,m)) - 1.$$  

(30)

(without loss of generality, matrix $P_m$ can be arbitrarily scaled).

The previous results extend to the HPLFs setting the results obtained in [8] by using QLFs, i.e., in the case $m = 1$. The next theorem clarifies this extension by showing that if the sufficient condition of Theorem 1 is satisfied for $m = 1$, then it is also for any $m > 1$. 

Theorem 2 If there exist $P_1 \in \mathbb{R}^{n \times n}$, $P_1 = P'_1 > 0$, $G_{1,E} \in \mathbb{R}^{n \times d}$ and $H_{1,E} \in \mathbb{R}^{d \times d}$ satisfying condition (26) of Theorem 1 for $m = 1$, then there exist $P_m \in \mathbb{R}^{\sigma(n,m) \times \sigma(n,m)}$, $P_m = P'_m > 0$, $G_{m,E} \in \mathbb{R}^{\sigma(n,m) \times d}$, $H_{m,E} \in \mathbb{R}^{d \times (n,m-1)}$ and $\beta_E = 0$ satisfying the same condition for any $m > 1$.

Proof Since $\mathcal{N}_1$ is empty, it turns out that for $m = 1$ condition (26) reduces to $R_1(P_1, G_{1,E}, H_{1,E}, E) < 0$. Hence, since $\beta_E = 0$, it is sufficient to show that also $R_m(P_m, G_{m,E}, H_{m,E}, E) < 0$ for $m > 1$.

Let us suppose $m = 2$ for simplicity, and let us select $P_2 = K_2 P'_1 K_2$. Since $K_2$ has full column rank, and, by assumption, $P_1 > 0$, it follows that $P_2 > 0$. Let us select $G_{2,E} = 2K_2(G_{1,E} \otimes P_1)$ and $H_{2,E} = 2H_{1,E} \otimes P_1$. Then, from Lemma 2 it turns out that

$$R_2(P_2, G_{2,E}, H_{2,E}, E) = 2X$$

where

$$X = \text{diag} (K_2, I_{dn})^T (R_1(P_1, G_{1,E}, H_{1,E}, E) \otimes P_1) \text{diag} (K_2, I_{dn}).$$

Since it has been assumed $R_1(P_1, G_{1,E}, H_{1,E}, E) < 0$, it follows that $R_2(P_2, G_{2,E}, H_{2,E}, E) < 0$, that is the sufficient condition of Theorem 1 is satisfied also for $m = 2$. The same reasoning can be repeated for $m > 2$.

The proof of Theorem 2 makes it clear that the same conclusion can be drawn for any choice of matrices $G_{m,E}$ and $H_{m,E}$, and therefore also for Corollary 1 and Corollary 2.

We finally mention that results analogous to Theorem 1, Corollary 1, and Corollary 2 can be derived also when bounds on the rate of variation of $\theta(t)$ are available. Details will be reported in future works.

3.2 Sufficient conditions for problem P2

The aim of this section is to show how Corollary 1 and Corollary 2 can be exploited for computing a lower bound of the robust stability margin $\gamma_{2m}$ through a GEVP, i.e., a quasi-convex optimization with LMI constraints (see [10] for details about GEVP).
Corollary 3  Let $\hat{\gamma}_{2m}$ be defined as

$$ (\hat{\gamma}_{2m})^{-1} = \min_{t \in \mathbb{R}, P_m, G_m, H_m, \beta_0, \ldots, \beta_l} t $$

s.t. \begin{align*}
   P_m &> 0 \\
   -R_m(P_m, G_m, H_m, 0) - N_m(\beta_0) &> 0 \\
   t(-R_m(P_m, G_m, H_m, 0) - N_m(\beta_0)) &> R_m(P_m, G_m, H_m, E_i) - R_m(P_m, G_m, H_m, 0) + N_m(\beta_i), \ i = 1, \ldots, l.
\end{align*}

Then, $\hat{\gamma}_{2m} \leq \gamma_{2m}$.

Proof  The result follows from Corollary 1, because for any $\gamma \leq \hat{\gamma}_{2m}$ and $i = 1, \ldots, l$ it turns out that

$$ R_m(P_m, G_m, H_m, \gamma E_i) + N_m(\gamma(\beta_i + \beta_0)) = \gamma(R_m(P_m, G_m, H_m, E_i) - R_m(P_m, G_m, H_m, 0) + N_m(\beta_i)) + R_m(P_m, G_m, H_m, 0) + N_m(\beta_0) < 0. $$

Observe also that condition $-R_m(P_m, G_m, H_m, 0) - N_m(\beta_0) > 0$ for some $\beta_0$ is satisfied implicitly in Corollary 1, since $E$ includes the origin.

Corollary 4  Suppose that $d_{LFR} = 1$. Let $\hat{\gamma}_{2m}^\#$ be defined as

$$ (\hat{\gamma}_{2m}^\#)^{-1} = \min_{t \in \mathbb{R}, P_m, G_m, i, H_m, \beta_{1,i}, \beta_{2,i}, i = 1, \ldots, l} t $$

s.t. \begin{align*}
   P_m &> 0 \\
   -R_m(P_m, G_m, i, H_m, 0) - N_m(\beta_{1,i}) &> 0, \ i = 1, \ldots, l \\
   t(-R_m(P_m, G_m, i, H_m, 0) - N_m(\beta_{1,i})) &> R_m(P_m, G_m, i, H_m, E_i) + \\
   &-R_m(P_m, G_m, i, H_m, 0) + N_m(\beta_{2,i}), \ i = 1, \ldots, l.
\end{align*}

Then, $\hat{\gamma}_{2m}^\# \leq \gamma_{2m}$.

Proof  From Corollary 2 and arguments analogous to the ones in the proof of Corollary 3.

The GEVPs in Corollary 3 and Corollary 4 involve respectively $\xi_1(n, d, m, l)$ and $\xi_2(n, d, m, l)$ free parameters, where

$$ \begin{align*}
   \xi_1(n, d, m, l) &= \zeta_1(n, d, m, l) + \tau_{par}(n, d, m) + 1 \\
   \xi_2(n, d, m, l) &= \zeta_2(n, d, m, l) + l\tau_{par}(n, d, m) + 1.
\end{align*} $$

(33)
(the top-left entry of $P_m$ has been supposed fixed).

Finally, we note that $\dot{\gamma}_{2m}$ in Corollary 3 and $\dot{\gamma}_{2m}^\#$ in Corollary 4 can be also computed via a bisection search on a scalar $\eta$ where, at each step, the $2m$-homogeneous stability of (3)-(4) over $\eta\Theta$ is ensured via the conditions in Corollary 1 and Corollary 2, respectively. This alternative procedure involves a smaller number of parameters in the LMI optimizations (see (33)).

4 Numerical examples

In this section, two examples are presented in order to illustrate the proposed approach. The first one shows that, as expected, better robust stability margins are obtained as the degree $m$ of the HPLF is increased. The second example shows a case in which the proposed technique produces significantly less conservative results, with respect to computing a polytope $\mathcal{A}$ bounding the uncertainty set $\{A_{rat}(\theta) : \theta \in \Theta\}$ and then employing the techniques developed in [7] for constructing a HPLF for polytope $\mathcal{A}$.

4.1 Example 1

Let us consider system (1) where

$$A_{rat}(\theta(t)) = \begin{bmatrix} 0 & 0 \\ \theta_1(t)\theta_2(t) + \theta_1(t) + \theta_2(t) + 3 & -\theta_1(t) - 1 \end{bmatrix}$$

and $\Theta = \text{co}\{[0,0]', [1,0]', [1,1]'\}$.

It turns out that (34) admits the LFR (3)-(4) where

$$A = \begin{bmatrix} 0 & -1 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E(\theta) = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}$$

(35)

$n = 2$, $d = 2$ and $d_{LFR} = 1$.

We are interested in computing a lower bound of the robust stability margin $\gamma_{2m}$ for $m = 1, 2, 3$.

Table 2 shows the lower bound $\hat{\gamma}_{2m}$ defined in Corollary 3 and the corresponding GEVP sizes. The same lower bounds are found from Corollary 4, i.e., the information that $d_{LFR} = 1$ does not allow us to obtain less conservative results in this example.

For completeness, we have considered the problem of establishing robust stability of (34) over $\dot{\gamma}_{6}\Theta$ by using the techniques for constructing HPLFs in [7]. Let us define $A_{rat}(a) = \{A_{rat}(\theta) : \theta \in a\Theta\}$ and $C_{rat}(a) = \{[c_1, c_2]' : c_1 = \theta_1\theta_2 + \theta_1 + \theta_2 + 3, c_2 = -\theta_1 - 1, \theta \in a\Theta\}$. The polytope $\mathcal{A}(\dot{\gamma}_6)$ bounding
set $A_{rat}(\gamma_6)$ is obtained as $A(\gamma_6) = \{[0,-1; c_1, c_2] : [c_1, c_2]' \in C(\gamma_6)\}$ where $C(\gamma_6)$ is the polytope bounding set $C_{rat}(\gamma_6)$ (see Figure 1). We have found that there exists a HPLF of degree 6 establishing robust stability of $\dot{x}(t) = A(t)x(t)$ for $A(t) \in A(\gamma_6)$ and, hence, robust stability of (34) over $\gamma_6\Theta$. In this case, hence, the conservative approximation of $\gamma_6\Theta$ with a polytope still allows us to establish robust stability by using a HPLF of degree 6.

\[
\begin{array}{|c|c|c|}
\hline
m & \hat{\gamma}_{2m} & \tau(2,2,m) \\
\hline
1 & 2.2351 & 4 \\
2 & 3.3998 & 7 \\
3 & 3.5708 & 10 \\
\hline
\end{array}
\]

Table 2: (Example 1) Lower bound $\hat{\gamma}_{2m}$ in Corollary 3 and corresponding GEVP sizes $\tau(2,2,m)$ (size of matrix $N_m(\cdot)$) and $\zeta_1(2,2,m,3)$ (number of free parameters).

\[
A_{rat}(\theta(t)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4\theta_2(t) - 4 & -3 & \frac{4\theta_1(t)\theta_2(t) + 7\theta_1(t) - 12}{4 - \theta_1(t)} \end{bmatrix}
\]

4.2 Example 2

Let us consider system (1) where

\[
A_{rat}(\theta(t)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4\theta_2(t) - 4 & -3 & \frac{4\theta_1(t)\theta_2(t) + 7\theta_1(t) - 12}{4 - \theta_1(t)} \end{bmatrix}
\]
and $\Theta = \text{co}\{[0,0]', [1,1]', [1,-1]']$. 

It turns out that (36) admits the LFR (3)-(4) where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \end{bmatrix}, \quad E(\theta) = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix},$$  

(37)

$n = 3$, $d = 2$ and $d_{LFR} = 1$.

Table 3 shows the lower bounds $\tilde{\gamma}_{2m}$ and $\tilde{\gamma}_2^{#}$ defined respectively in Corollaries 3 and 4, and the corresponding GEVP sizes for some values of $m$.

Analogously to the previous example, we have considered the problem of establishing robust stability of (36) over $\tilde{\gamma}_2^{#}, \Theta, i = 1, 2, 3$, by using the techniques for constructing HPLFs in [7]. Let us define $C_{rat}(a) = \{[c_1, c_2]' : c_1 = 4\theta_2 - 4, c_2 = (4\theta_1\theta_2 + 7\theta_1 - 12)/(4 - \theta_1), \theta \in a\Theta\}$. The polytope $A(\tilde{\gamma}_2^{#})$ bounding set $A_{rat}(\tilde{\gamma}_2^{#})$ is obtained as $A(\tilde{\gamma}_2^{#}) = \{(0,1,0;0,0;1;c_1,-3,c_2) : [c_1,c_2]' \in C(\tilde{\gamma}_2^{#})\}$ where $C(\tilde{\gamma}_2^{#})$ is the polytope bounding set $C_{rat}(\tilde{\gamma}_2^{#})$ (see Figure 2). We have found that robust stability of $\dot{x}(t) = A(t)x(t)$ for $A(t) \in \mathcal{A}(\tilde{\gamma}_2^{#})$ cannot be proved by using a HPLF of degree 2 but can be proved by using a HPLF of degree 4, and that robust stability for $A(t) \in \mathcal{A}(\tilde{\gamma}_4^{#})$ cannot be proved not only by using a HPLF of degree 4 but also by using a HPLF of degree 12. In particular, by using a HPLF of degree 12, the robust stability margin $\tilde{\gamma}$ that can be guaranteed for $A(t) \in \mathcal{A}(\tilde{\gamma})$ is 0.8106.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\tilde{\gamma}_{2m}$</th>
<th>$\tau(3,2,m)$</th>
<th>$\zeta_1(3,2,m,3)$</th>
<th>$m$</th>
<th>$\tilde{\gamma}_2^{#}$</th>
<th>$\tau(3,2,m)$</th>
<th>$\zeta_2(3,2,m,3)$</th>
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</thead>
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<tr>
<td>1</td>
<td>0.55342</td>
<td>5</td>
<td>16</td>
<td>1</td>
<td>0.57894</td>
<td>5</td>
<td>36</td>
</tr>
<tr>
<td>2</td>
<td>0.77526</td>
<td>12</td>
<td>193</td>
<td>2</td>
<td>0.81157</td>
<td>12</td>
<td>387</td>
</tr>
<tr>
<td>3</td>
<td>0.84516</td>
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<td>871</td>
<td>3</td>
<td>0.87758</td>
<td>22</td>
<td>1675</td>
</tr>
</tbody>
</table>

Table 3: (Example 2) Lower bounds $\tilde{\gamma}_{2m}$ (a) and $\tilde{\gamma}_2^{#}$ (b). The corresponding GEVP sizes are $\tau(3,2,m)$ (size of matrix $N_m(\cdot)$), $\zeta_1(3,2,m,3)$ and $\zeta_2(3,2,m,3)$ (number of free parameters).
5 Conclusion

Robust stability of linear systems subject to time-varying parametric uncertainty, affecting rationally the state transition matrix, has been considered. Sufficient conditions based on LMI feasibility tests have been derived for the existence of HPLFs ensuring robust stability, when the uncertain parameter vector is restricted to lie in some polytope. LMI-based conditions for computing lower bounds of robust stability margins have been also given. The conditions involving HPLFs have been proven to provide less conservative results than those generated via QLFs. Some numerical examples have been developed to illustrate the features of the approach and to show that the gap between HPLFs and QLFs can be significantly large.

Finally, we mention that some other relevant issues, e.g., the presence of bounds on the rate of variation of the uncertain parameter vector, or the computation of the HPLF ensuring the best transient performance, can be easily incorporated in the considered framework.

A Appendix

A.1 Proof of Lemma 1

Set $\mathcal{N}_m$ is a linear space since if $X_1, X_2 \in \mathcal{N}_m$ then $x_1X_1 + x_2X_2 \in \mathcal{N}_m$ for any $x_1, x_2 \in \mathbb{R}$. Let us consider the dimension of $\mathcal{N}_m$. The first term in (15) is the number of distinct entries of a $\tau(n, d, m) \times \tau(n, d, m)$ symmetric matrix, while the sum of the three terms in brackets is the number of distinct monomials in the homogeneous form $y_m(x; p)N_my_m(x; p)$, of degree $2m$ in $(x; p)$, for any $\mathcal{N}_m$. Since the coefficients of the homogeneous form depend linearly on the entries of $N_m$, it follows that the dimension of $\mathcal{N}_m$ is given by (15).

A.2 Parameterization of $\mathcal{N}_m$

According to Lemma 1, there exist matrices $N_{m,1}, N_{m,2}, \ldots, N_{m,\tau_{par}(n,d,m)}$ forming a base of the linear space $\mathcal{N}_m$. In the following, it will be shown how to compute such matrices.

Let $b$ be a vector of monomials; for any element $b_i$ of $b$ define the operator

$$\text{ind}\{b_i, b\} = \text{position of element } b_i \text{ in the vector } b.$$ 

Define the vectors

$$y_m(x; p) = \left[ \begin{array}{c} x^{\{m\}} \\ p \otimes x^{\{m-1\}} \end{array} \right] \in \mathbb{R}^{\tau(n,d,m)}$$
and

\[
\varepsilon_{2m}(x; p) = \begin{bmatrix} x^{(2m)}_p & x^{(2m-1)}_p \\ p \otimes x^{(2m-1)} & p^{(2)} \otimes x^{(2m-2)} \end{bmatrix} \in \mathbb{R}^{\phi(n,d,m)}
\]

where

\[
\phi(n, d, m) = \sigma(n, 2m) + d\sigma(n, 2m - 1) + \frac{d(d + 1)}{2}\sigma(n, 2m - 2).
\]

An algorithm for the computation of \(N_m,1, N_m,2, \ldots, N_m,\tau_{par}(n,d,m)\) is reported in Table 4, where \([N_m,h]_{i,j}\) denotes the element \((i, j)\) of matrix \([N_m,h]\).

### A.3 Proof of Lemma 2

Let us consider (20). From (17) and (24) it follows that

\[
K_m \tilde{A}_m(A) x^{(m)} = \frac{\partial x^{(m)}}{\partial x} Ax = \left( \sum_{j=0}^{m-1} x[j] \otimes I_n \otimes x^{[m-1-j]} \right) Ax.
\]

Then, (20) follows by observing that, for any \(j = 0, \ldots, m - 1\),

\[
(x[j] \otimes I_n \otimes x^{[m-1-j]}) Ax = x[j] \otimes Ax \otimes x^{[m-1-j]} = (I_{n^j} \otimes A \otimes I_{n^{m-1-j}}) x^{[m]} = (I_{n^j} \otimes A \otimes I_{n^{m-1-j}}) K_m x^{(m)}.
\]

Let us consider (21). From (17) and (24) it follows that

\[
K_m B_m(B) (p \otimes x^{(m-1)}) = \frac{\partial x^{[m]}}{\partial x} Bp = \left( \sum_{j=0}^{m-1} x[j] \otimes I_n \otimes x^{[m-1-j]} \right) Bp.
\]

For any \(j = 0, \ldots, m - 1\) it turns out that

\[
(x[j] \otimes I_n \otimes x^{[m-1-j]}) Bp = x[j] \otimes Bp \otimes x^{[m-1-j]} = (I_{n^j} \otimes B \otimes I_{n^{m-1-j}}) (x[j] \otimes p \otimes x^{[m-1-j]})
\]

\[
= (I_{n^j} \otimes B \otimes I_{n^{m-1-j}}) (F_j (p \otimes x[j]) \otimes x^{[m-1-j]})
\]

\[
= (I_{n^j} \otimes B \otimes I_{n^{m-1-j}}) (F_j \otimes I_{n^{m-1-j}}) (p \otimes x^{[m-1]}).
\]
Then, (21) follows by observing that
\[ p \otimes x^{[m-1]} = (I_d \otimes K_{m-1}) \left( p \otimes x^{[m-1]} \right). \]  

(38)

Let us consider (22). From (18), (24) and (38) it follows that
\[
(I_d \otimes K_{m-1}) \tilde{C}_m(C)x^{[m]} = Cx \otimes x^{[m-1]} \\
= (C \otimes I_{n^{m-1}}) x^{[m]} \\
= (C \otimes I_{n^{m-1}}) K_m x^{[m]}.
\]

Therefore,
\[
\tilde{C}_m(C) = \left[ (I_d \otimes K'_m) (I_d \otimes K_{m-1}) \right]^{-1} (I_d \otimes K'_m) (C \otimes I_{n^{m-1}}) K_m \\
= \left( I_d \otimes (K'_m K_{m-1})^{-1} \right) (I_d \otimes K'_m) (C \otimes I_{n^{m-1}}) K_m \\
= \left( C \otimes (K'_m K_{m-1})^{-1} K'_m \right) K_m.
\]

Finally, (23) follows directly from (17) and Kronecker’s product’s properties.

References


Figure 2: (Example 2). Sets $C_{\text{rat}}(\gamma_{2i}^\#)$ (dashed regions) and boundary of polytopes $C(\gamma_{2i}^\#), i = 1, 2, 3$. 
Algorithm for the computation of $N_{m,1}, N_{m,2}, \ldots, N_{m,\tau_{par}(n,d,m)}$

1. $V = 0_{\phi(n,d,m) \times 3}$
2. $q = 0$
3. for $i = 1, \ldots, \tau(n,d,m)$ and $j = i, \ldots, \tau(n,d,m)$
   4. $k = \text{ind}\left\{ y_m(x;p), y_m(x;p)_j, z_{2m}(x;p) \right\}$
   5. $V_{k,1} = V_{k,1} + 1$
   6. if $V_{k,1} = 1$
   7. $V_{k,2:3} = (i, j)$
   8. else
   9. $q = q + 1$
10. $N_{m,q} = 0_{\tau(n,d,m) \times \tau(n,d,m)}$
11. $[N_{m,q}]_{i,j} = [N_{m,q}]_{i,j} + 1$
12. $[N_{m,q}]_{j,i} = [N_{m,q}]_{j,i} + 1$
13. $\tilde{i} = V_{k,2}$
14. $\tilde{j} = V_{k,3}$
15. $[N_{m,q}]_{\tilde{i},\tilde{j}} = [N_{m,q}]_{i,j} - 1$
16. $[N_{m,q}]_{\tilde{j},\tilde{i}} = [N_{m,q}]_{j,i} - 1$
17. endif
18. endfor

Table 4: Algorithm for the computation of $N_{m,1}, N_{m,2}, \ldots, N_{m,\tau_{par}(n,d,m)}$. 