Polynomially parameter-dependent Lyapunov functions for robust stability of polytopic systems: an LMI approach

G. Chesi¹, A. Garulli¹, A. Tesi², A. Vicino¹

¹Dipartimento di Ingegneria dell’Informazione, Università di Siena
Via Roma 56, 53100 Siena, Italy
E-mail: {chesi,garulli,vicino}@dii.unisi.it

²Dipartimento di Sistemi e Informatica, Università di Firenze
Via di S.Marta 3, 50139 Firenze, Italy
E-mail: atesi@dsi.unifi.it

Abstract

In this paper robust stability of state space models with respect to real parametric uncertainty is considered. Specifically, a new class of parameter-dependent quadratic Lyapunov functions for establishing stability of a polytope of matrices is introduced, i.e., the Homogeneous Polynomially Parameter-Dependent Quadratic Lyapunov Functions (HPD-QLFs). The choice of this class, which contains parameter-dependent quadratic Lyapunov functions whose dependence on the uncertain parameters is expressed as a polynomial homogeneous form, is motivated by the property that a polytope of matrices is stable if and only there exists a HPD-QLF. The main result of the paper is a sufficient condition for determining the sought HPD-QLF, which amounts to solving Linear Matrix Inequalities (LMIs) derived via the Complete Square Matricial Representation (CSMR) of homogeneous matricial forms and the Lyapunov matrix equation. Numerical examples are provided to demonstrate the effectiveness of the proposed approach.

I. INTRODUCTION

Robust stability of state space models subject to time-invariant parametric uncertainty has received considerable attention since long time [1], [2]. Within this context, a fundamental problem is that of establishing whether a polytope of matrices contains only Hurwitz matrices
(continuous-time case) or Schur matrices (discrete-time case). Although several computational procedures have been proposed to address this problem, completely satisfactory results have been obtained for special classes of matrices only (see [3], [4] and references therein). On the other hand, it is well known that the computational complexity reduction in the case of polytope of polynomials does not pertain to the case of polytope of matrices [5], and therefore only sufficient conditions are available for assessing stability of general polytopes of matrices.

Most of these conditions have been obtained via Lyapunov based approaches. The simplest approach consists in looking for a common quadratic Lyapunov function that proves stability of the polytope of matrices (see e.g. [6]). Such a problem is quite appealing from a computational point of view since it amounts to solving Linear Matrix Inequalities (LMIs) that are special convex optimizations [7]. On the other hand, it is well known that the existence of a common Lyapunov function is strictly related to the circle criterion of absolute stability theory [8], and therefore cases where the results are quite conservative can be easily found. In order to provide less conservative results, parameter-dependent quadratic Lyapunov functions have been employed (see e.g. [3, pp. 342-347]). A characterization result has been given in [9], where it has been shown that linearly parameter-dependent quadratic Lyapunov functions are necessary and sufficient for robust stability of some classes of systems with rank-one uncertainty. Recently, several techniques involving parameter-dependent quadratic Lyapunov functions have been proposed, which provide sufficient conditions resulting in LMIs ([10], [11], [12], [13], [14], [15]). Finally, for the special case where the polytope is a hypercube, a necessary and sufficient LMI condition based on the use of a quadratic Lyapunov function depending polynomially on the uncertain parameters has been given in [16].

In this paper the use of a new class of parameter-dependent quadratic Lyapunov functions for establishing stability of a polytope of matrices is investigated, i.e., the Homogeneous Polynomially Parameter-Dependent Quadratic Lyapunov Functions (simply abbreviated as HPD-QLFs). This is the class of parameter-dependent quadratic Lyapunov functions, whose dependence on the uncertain parameters is expressed as a polynomial homogeneous form. The main motivation for selecting this class, which is a natural generalization of linearly parameter-dependent QLFs, is that it is rich enough to provide a complete answer to the studied problem. Specifically, it turns out that a polytope of matrices is stable if and only if there exists a HPD-QLF. It is worth remarking that the potential of homogeneous polynomial forms for the analysis of control systems
has been recognized since long time (see e.g., [17], [18]). In recent years, homogeneous forms gained a renewed interest, motivated by the strong connection with semidefinite programming and convex optimization techniques [19], which made it possible to exploit them for solving several problems (see e.g. [20]).

The main contribution of the paper is a sufficient condition to determine the sought HPD-QLF. Such a condition amounts to solving LMIs which are derived by using the Complete Square Matricial Representation (CSMR) of homogeneous matricial forms and the Lyapunov matrix equation. The case of continuous-time systems is treated in detail, but the extension to the discrete-time case can be easily accomplished. Numerical examples are provided to compare the effectiveness of the proposed approach, with respect to methods based on linearly parameter-dependent Lyapunov functions.

II. PROBLEM FORMULATION AND PRELIMINARIES

Notation:

- \(0_n, 0_{m \times n}\): origin of \(\mathbb{R}^n\) and of \(\mathbb{R}^{m \times n}\);
- \(\mathbb{R}^n_0\): \(\mathbb{R}^n \setminus \{0_n\}\);
- \(I_n\): identity matrix \(n \times n\);
- \(A'\): transpose of matrix \(A\);
- \(A > 0 (A \geq 0)\): symmetric positive definite (semidefinite) matrix \(A\);
- \(A \otimes B\): Kronecker’s product of matrices \(A\) and \(B\);
- \(\text{sv}([p_1, p_2, \ldots, p_q]') = [p_1^2, p_2^2, \ldots, p_q^2]'\).

Consider the continuous-time state space model

\[
\dot{x}(t) = A(p)x(t),
\]

where \(x \in \mathbb{R}^n\) is the state vector, and \(p = [p_1, p_2, \ldots, p_q]' \in \mathbb{R}^q\) is the uncertain parameter vector which belongs to the set

\[
\mathcal{P} = \left\{ p \in \mathbb{R}^q : \sum_{i=1}^{q} p_i = 1, \quad p_i \geq 0, \quad i = 1, 2, \ldots, q \right\}.
\]

The matrix \(A(p)\) is assumed of the following structure

\[
A(p) = \sum_{i=1}^{q} p_i A_i,
\]
where $A_i \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, q$, are given real matrices. Consider the polytope of matrices defined as
\[ \mathcal{A} = \{ A(p) \in \mathbb{R}^{n \times n} : p \in \mathcal{P} \}. \] (4)

The problem we address can be stated as follows.

**Robust stability problem:**
Establish if the polytope $\mathcal{A}$ is Hurwitz, i.e., $\mathcal{A}$ contains only Hurwitz matrices.

The key step for addressing the above problem is the construction of a Homogeneous Polynomially Parameter-Dependent Quadratic Lyapunov Function (simply abbreviated as HPD-QLF)
\[ v_m(x; p) = x'P_m(p)x, \] (5)
where $P_m(p) \in \mathbb{R}^{n \times n}$ is a homogeneous matricial form of degree $m$, i.e., a matrix whose entries are (real $q$-variate) homogeneous forms of degree $m$.

The following important property pertains to the class of HPD-QLFs.

**Lemma 1:** The set $\mathcal{A}$ is Hurwitz if and only if there exists a HPD-QLF $v_m(x; p)$ such that
\[ \begin{cases} P_m(p) > 0 \\ A'(p)P_m(p) + P_m(p)A(p) < 0 \end{cases} \forall p \in \mathcal{P}. \] (6)

**Proof.** Sufficiency is obvious since if there exists $P_m(p)$ satisfying (6), then $\mathcal{A}$ is clearly Hurwitz. Regarding the necessity, let us suppose that $\mathcal{A}$ is Hurwitz. Let $E_{\bar{m}}(p) = E'_{\bar{m}}(p)$ be any homogeneous matricial form of degree $\bar{m}$ such that $E_{\bar{m}}(p) > 0 \forall p \in \mathcal{P}$, and let us consider the Lyapunov equation
\[ A'(p)P(p) + P(p)A(p) = -E_{\bar{m}}(p). \] (7)
The solution is a rational matricial function $P(p) = P'(p) > 0 \forall p \in \mathcal{P}$ with homogeneous numerators and denominators. Let us write $P(p)$ as $P(p) = d^{-1}(p)P_m(p)$ where $d(p)$ is the denominator of $P(p)$ satisfying $d(p) > 0 \forall p$. Then, it clearly follows that $P_m(p)$ satisfies (6) (in particular, $A'(p)P_m(p) + P_m(p)A(p) = -d(p)E_{\bar{m}}(p)$).

**Remark 1.** A result analogous to that of Lemma 1 can be easily obtained for discrete-time systems.
Remark 2. From the proof of Lemma 1, an upper bound on the degree $m$ of the homogeneous matricial form $P_m(p)$ defining the HPD-QLF can be derived. In particular, by choosing $E_m(p) = E_0$ constant, one has $m < \frac{1}{2}n(n + 1)$.

A. Parameterization of homogeneous matricial forms

In order to give sufficient conditions for the existence of a HPD-QLF, it is useful to introduce a suitable parameterization of homogeneous matricial forms.

First, let us recall the Complete Square Matricial Representation (CSMR) of homogeneous scalar forms, which provides all possible representations of a homogeneous polynomial form of degree $2m$ in terms of a quadratic form in the space of the monomials of degree $m$ (see [21] for details). Let $w_{2m}(p)$ be a homogeneous form of degree $2m$ in $p \in \mathbb{R}^q$. The CSMR of $w_{2m}(p)$ is defined as

$$w_{2m}(p) = p^{(m)'}(W_m + L_m(\alpha))p^{(m)}$$

where:
- $p^{(m)} \in \mathbb{R}^{\sigma(q,m)}$ is the vector containing all monomials of degree $m$ in $p$;
- $W_m \in \mathbb{R}^{\sigma(q,m) \times \sigma(q,m)}$ is a suitable symmetric matrix;
- $\alpha \in \mathbb{R}^{\sigma_{par}(q,m)}$ is a vector of free parameters;
- $L_m(\alpha)$ is a linear parameterization of the set
  $$\mathcal{L}_m = \left\{ L_m = L_m' : p^{(m)'}L_m p^{(m)} = 0 \quad \forall p \in \mathbb{R}^q \right\}.$$

The quantities $\sigma(q, m)$ and $\sigma_{par}(q, m)$ are given respectively by ([21])

$$\sigma(q, m) = \frac{(q + m - 1)!}{(q - 1)!m!},$$

$$\sigma_{par}(q, m) = \frac{1}{2}\sigma(q, m)[\sigma(q, m) + 1] - \sigma(q, 2m).$$

Similarly to what has been done for scalar forms, one can introduce the CSMR for homogeneous matricial forms. Let $C_{2m}(p) \in \mathbb{R}^{n \times n}$ be a homogeneous matricial form of degree $2m$ in $p \in \mathbb{R}^q$. Then, $C_{2m}(p)$ can be written as

$$C_{2m}(p) = (p^{(m)} \otimes I_n)' \bar{C}_m (p^{(m)} \otimes I_n)$$

(10)
$\bar{C}_m \in \mathbb{R}^{n\sigma(q,m) \times n\sigma(q,m)}$ is a suitable matrix (denoted hereafter as a SMR matrix of $C_{2m}(p)$). Such a matrix is not unique and, indeed, all the matrices $\bar{C}_m$ describing $C_{2m}(p)$ are given by

$$\bar{C}_m + \bar{U}_m, \quad \bar{U}_m \in \mathcal{U}_m$$  \hspace{1cm} (11)

where

$$\mathcal{U}_m = \left\{ \bar{U}_m = \bar{U}_m' \in \mathbb{R}^{n\sigma(q,m) \times n\sigma(q,m)} : \begin{pmatrix} p^{\{m\}} \otimes I_n \end{pmatrix}' \bar{U}_m \begin{pmatrix} p^{\{m\}} \otimes I_n \end{pmatrix} = 0_{n \times n} \quad \forall p \in \mathbb{R}^q \right\}.$$  \hspace{1cm} (12)

**Lemma 2**: The set $\mathcal{U}_m$ in (12) is a linear space of dimension

$$u(q, n, m) = \frac{1}{2} n \left\{ \sigma(q, m) [n\sigma(q, m) + 1] - (n + 1)\sigma(q, 2m) \right\}.$$  \hspace{1cm} (13)

**Proof**. Set $\mathcal{U}_m$ is a linear space since $\bar{Z}_1, \bar{Z}_2 \in \mathcal{U}_m \Rightarrow z_1 \bar{Z}_1 + z_2 \bar{Z}_2 \in \mathcal{U}_m \forall z_1, z_2 \in \mathbb{R}$. Now, let us observe that $n\sigma(q, m)(n\sigma(q, m) + 1)/2$ is the number of entries of a symmetric matrix of dimension $n\sigma(q, m) \times n\sigma(q, m)$, while $n(n + 1)\sigma(q, 2m)/2$ is the number of independent terms (and, hence, of constraints) of a homogeneous $n \times n$ matricial form of degree $2m$ in $q$ variables. 

Let $\bar{U}_m(\alpha), \alpha \in \mathbb{R}^{u(q,n,m)}$, be a linear parameterization of $\mathcal{U}_m$. The CSMR of $C_{2m}(p)$ is hence given by

$$C_{2m}(p) = \begin{pmatrix} p^{\{m\}} \otimes I_n \end{pmatrix}' \left( \bar{C}_m + \bar{U}_m(\alpha) \right) \begin{pmatrix} p^{\{m\}} \otimes I_n \end{pmatrix}.$$

\hspace{1cm} (14)

**III. ROBUST STABILITY ANALYSIS**

In this section, sufficient conditions for robust stability are provided in terms of LMIs. The aim is to find a HPD-QLF as in (5), such that $P_m(p)$ satisfies (6) in Lemma 1. The first condition to be satisfied is the positive definiteness of the HPD-QLF matrix $P_m(p)$ within the set $\mathcal{P}$, i.e. the first inequality in (6). In this respect, a parameterization of positive definite matrices $P_m(p)$ is provided next. The second inequality in (6) will be dealt with in Section III-A. The following result exploits a basic property of homogeneous forms to give an alternative characterization of the positivity of $P_m(p)$.

**Lemma 3**: The condition

$$P_m(p) > 0 \quad \forall p \in \mathcal{P}$$  \hspace{1cm} (15)

holds if and only if

$$P_m(sv(p)) > 0 \quad \forall p \in \mathbb{R}_0^q.$$  \hspace{1cm} (16)
Proof. Being $P_m(p)$ homogeneous in $p$, one has that (15) is equivalent to $P_m(kp) > 0$ for all $p \in \mathcal{P}$ and for all positive $k$, and hence to $P_m(p) > 0$ for all $p$ in the positive orthant (i.e., such that $p_i \geq 0, i = 1, \ldots, q$, and $p \neq 0_n$). The latter condition can be equivalently expressed as in (16). ■

**Remark 3.** Notice that Lemma 3 still holds if the parametric uncertainty region $\mathcal{P}$ is replaced by any set of the form $\{p : p_i \geq 0, i = 1, \ldots, q; \|p\|_\ell = \gamma\}$, for any norm $\|\cdot\|_\ell$ and positive $\gamma$.

Observe that $P_m(sv(p))$ can be written as

$$P_m(sv(p)) = (p^{(m)} \otimes I_n)' \bar{S}_m (p^{(m)} \otimes I_n)$$

for some suitable matrix $\bar{S}_m \in \mathcal{S}_m$ where

$$\mathcal{S}_m = \left\{ \bar{S}_m = \bar{S}_m' \in \mathbb{R}^{n\sigma(q,m) \times n\sigma(q,m)} : (p^{(m)} \otimes I_n)' \bar{S}_m (p^{(m)} \otimes I_n) \text{ does not contain entries } p_1^{i_1}p_2^{i_2} \ldots p_q^{i_q} \text{ with any odd } i_j \right\}.$$  

From the definition of $\mathcal{S}_m$, an alternative way to write (17) is

$$P_m(sv(p)) = \tilde{T}_m ([sv(p)]^{(m)} \otimes I_n)$$

where $\tilde{T}_m \in \mathbb{R}^{n \times n\sigma(q,m)}$ is a suitable matrix. Hence, due to Lemma 3, one has that if $\bar{S}_m$ in (17) is positive definite, then the matrix

$$P_m(p) = \tilde{T}_m (p^{(m)} \otimes I_n)$$

is positive definite for $p \in \mathcal{P}$.

In order to increase the degrees of freedom in the selection of $P_m(p)$, it is worth noticing that matrix $\bar{S}_m$ in (17) is not unique. The next lemma provides a characterization of the set $\mathcal{S}_m$.

**Lemma 4:** The set $\mathcal{S}_m$ is a linear space of dimension

$$s(q,n,m) = \frac{1}{2} n \left\{ \sigma(q,m) [n\sigma(q,m) + 1] - (n + 1)[\sigma(q,2m) - \sigma(q,m)] \right\}.$$  

Proof. The set $\mathcal{S}_m$ is a linear space since $\bar{Z}_1, \bar{Z}_2 \in \mathcal{S}_m \Rightarrow z_1 \bar{Z}_1 + z_2 \bar{Z}_2 \in \mathcal{S}_m \forall z_1, z_2 \in \mathbb{R}$. Now, let us observe that $n\sigma(q,m)(n\sigma(q,m) + 1)/2$ is the number of entries of a symmetric matrix of dimension $n\sigma(q,m) \times n\sigma(q,m)$, while $n(n + 1)(\sigma(q,2m) - \sigma(q,m))/2$ is the number of independent terms (and, hence, of constraints) containing at least one odd power of a homogeneous
Let $\bar{S}_m(\beta)$, $\beta \in \mathbb{R}^{s(q,n,m)}$, be a linear parameterization of $S_m$. Clearly, this induces a corresponding linear parameterization $\bar{T}_m(\beta)$ of matrix $\bar{T}_m$ in (19). Hence, one can choose the family of candidate HPD-QLF matrices

$$P_m(p; \beta) = \bar{T}_m(\beta) \left( p^{\{m\}} \otimes I_n \right)$$

which depends linearly on the parameterization $\beta$ of $S_m$. Following the above reasoning, one has the next result, which is the key step for the formulation of the sufficient condition for solving the robust stability problem.

**Lemma 5:** Let $\bar{S}_m(\beta)$ belong to $S_m$ in (18). If $\bar{S}_m(\beta) > 0$, then

$$P_m(p; \beta) > 0 \ \forall p \in \mathcal{P}.$$  

**A. LMI-based sufficient conditions**

In the following, a sufficient condition for the solution of the robust stability problem is provided. To this purpose, let us introduce the homogeneous matricial form of degree $m + 1$

$$Q_{m+1}(p; \beta) = -A'(p)P_m(p; \beta) - P_m(p; \beta)A(p)$$

and the related homogeneous form $Q_{m+1}(sv(p); \beta)$, which can be written as

$$Q_{m+1}(sv(p); \beta) = (p^{\{m+1\}} \otimes I_n)' \bar{R}_{m+1}(\beta) (p^{\{m+1\}} \otimes I_n),$$

where $\bar{R}_{m+1}(\beta) \in \mathbb{R}^{n\sigma(q,m+1) \times n\sigma(q,m+1)}$ is any SMR matrix of $Q_{m+1}(sv(p); \beta)$. The following result yields the sought sufficient condition.

**Theorem 1:** The polytope $A$ in (4) is Hurwitz if there exist a nonnegative integer $m$, and parameter vectors $\alpha \in \mathbb{R}^{u(q,n,m+1)}$ and $\beta \in \mathbb{R}^{s(q,n,m)}$ such that

$$\begin{cases}
\bar{S}_m(\beta) > 0 \\
\bar{R}_{m+1}(\beta) + \bar{U}_{m+1}(\alpha) > 0
\end{cases}$$

where $\bar{S}_m(\beta) \in S_m$, $\bar{U}_{m+1}(\alpha) \in U_{m+1}$, and $\bar{R}_{m+1}(\beta)$ is defined by (22)-(23).

**Proof.** First, let $P_m(p; \beta)$ be defined as in (21). Then, from (24) and Lemma 5 one has that $P_m(p; \beta) > 0 \ \forall p \in \mathcal{P}$, and hence the first condition in (6) holds. Second, let us observe that
\( \bar{R}_{m+1}(\beta) + \bar{U}_{m+1}(\alpha) \) is the CSMR matrix of \( Q_{m+1}(sv(p); \beta) \) in (23). Hence, (24) implies that \( Q_{m+1}(sv(p); \beta) > 0 \ \forall p \in \mathbb{R}_0^q \). From Lemma 3 it turns out that \( Q_{m+1}(p; \beta) > 0 \ \forall p \in \mathcal{P} \) and, therefore, \( \mathcal{A} \) is Hurwitz.

The inequalities (24) form an LMI feasibility problem with \( s(q, n, m) + u(q, n, m + 1) \) free parameters. The size of the matrices is \( n \sigma(q, m) \) for the first inequality and \( n \sigma(q, m + 1) \) for the second one. The solution can be computed by using efficient convex optimization tools, like [22], [23].

A question that naturally arises is whether there exists a relationship between the families of HPD-QLFs of degree \( m \) and \( m + 1 \). The following result clarifies that, if the sufficient condition of Theorem 1 is satisfied for \( m \), then it is satisfied also for \( m + 1 \).

**Theorem 2:** Let \( m \) be a nonnegative integer. If there exist parameter vectors \( \alpha \in \mathbb{R}^{u(q,n,m+1)} \) and \( \beta \in \mathbb{R}^{s(q,n,m)} \) such that (24) is satisfied, then there exist parameter vectors \( \tilde{\alpha} \in \mathbb{R}^{u(q,n,m+2)} \) and \( \tilde{\beta} \in \mathbb{R}^{s(q,n,m+1)} \) such that

\[
\begin{align*}
\bar{S}_{m+1}(\tilde{\beta}) &> 0 \\
\bar{R}_{m+2}(\tilde{\beta}) + \bar{U}_{m+2}(\tilde{\alpha}) &> 0
\end{align*}
\]

(25)

**Proof.** From the proof of Theorem 1 we have that, \( \forall p \in \mathcal{P} \), \( P_m(p; \beta) > 0 \) and \( Q_{m+1}(p; \beta) > 0 \). Define now \( P_{m+1}(p) = P_m(p; \beta) \sum_{i=1}^q p_i \). It follows that \( P_{m+1}(p) > 0 \ \forall p \in \mathcal{P} \) and \( Q_{m+2}(p) = Q_{m+1}(p; \beta) \sum_{i=1}^q p_i > 0 \ \forall p \in \mathcal{P} \). This means that \( v_{m+1}(x; p) = x'P_{m+1}(p)x \) is a HPD-QLF of degree \( m + 1 \) satisfying the condition of Lemma 1.

Let us show now that \( P_{m+1}(sv(p)) \) admits a positive definite SMR matrix, that is there exists \( \tilde{\beta} \) such that \( \bar{S}_{m+1}(\tilde{\beta}) > 0 \). Let \( K_{m+1} \) be the matrix satisfying

\[
p \otimes p^{(m)} = K_{m+1}p^{(m+1)} \ \forall p \in \mathbb{R}^q.
\]
Then,
\[
\begin{align*}
P_{m+1}(sv(p)) &= (\sum_{i=1}^{q} p_i^2) (p^{(m)} \otimes I_n)' \bar{S}_m(\beta) (p^{(m)} \otimes I_n) \\
&= p'p (p^{(m)} \otimes I_n)' \bar{S}_m(\beta) (p^{(m)} \otimes I_n) \\
&= (p \otimes p^{(m)} \otimes I_n)' (I_q \otimes \bar{S}_m(\beta)) (p \otimes p^{(m)} \otimes I_n) \\
&= (K_{m+1} p^{(m+1)} \otimes I_n)' (I_q \otimes \bar{S}_m(\beta)) (K_{m+1} p^{(m+1)} \otimes I_n) \\
&= (p^{(m+1)} \otimes I_n)' (K_{m+1} \otimes I_n)' (I_q \otimes \bar{S}_m(\beta)) (K_{m+1} \otimes I_n) (p^{(m+1)} \otimes I_n) \\
&= (p^{(m+1)} \otimes I_n)' \bar{S}_{m+1} (p^{(m+1)} \otimes I_n)
\end{align*}
\]
where
\[
\bar{S}_{m+1} = (K_{m+1} \otimes I_n)' (I_q \otimes \bar{S}_m(\beta)) (K_{m+1} \otimes I_n).
\]

From (26), it is clear that \( \bar{S}_{m+1} \in S_{m+1} \), and hence there exists \( \bar{\beta} \) such that \( \bar{S}_{m+1}(\bar{\beta}) = \bar{S}_{m+1} \).

Moreover, since \( \bar{S}_m(\beta) > 0 \) and \( K_{m+1} \) is a matrix with full column rank, it follows that \( \bar{S}_{m+1}(\bar{\beta}) > 0 \).

Let us show now that \( Q_{m+2}(sv(p)) \) admits a positive definite SMR matrix. Following the same development as in (26), one gets

\[
Q_{m+2}(sv(p)) = (p^{(m+2)} \otimes I_n)' \bar{R}_{m+2}(\bar{\beta}) (p^{(m+2)} \otimes I_n)
\]

(27)

where
\[
\bar{R}_{m+2}(\bar{\beta}) = (K_{m+2} \otimes I_n)' (I_q \otimes (\bar{R}_{m+2}(\beta) + \bar{U}_{m+1}(\alpha))) (K_{m+2} \otimes I_n)
\]

(28)

Since \( \bar{R}_{m+1}(\beta) + \bar{U}_{m+1}(\alpha) > 0 \) it follows that \( \bar{R}_{m+2}(\bar{\beta}) > 0 \). Therefore, \( Q_{m+2}(sv(p)) \) admits the positive definite SMR matrix \( \bar{R}_{m+2}(\bar{\beta}) + \bar{U}_{m+2}(\bar{\alpha}) \) with \( \bar{\alpha} = 0_{u(q,n,m+2)} \), and (25) holds. \( \blacksquare \)

Remark 4. The proposed technique can be applied also to discrete-time systems \( x(t+1) = A(p)x(t) \), where \( A(p) \) belongs to the polytope in (4). An LMI-based sufficient condition similar to (24) can be obtained by observing that

\[
P_m(p) - A'(p)P_m(p)A(p) = P_m(p) \left( \sum_{i=1}^{q} p_i \right)^2 - A'(p)P_m(p)A(p)
\]

and the right term is a homogeneous matricial form of degree \( m + 2 \) that can be parameterized as

\[
Q_{m+2}(p; \beta) = P_m(p; \beta) \left( \sum_{i=1}^{q} p_i \right)^2 - A'(p)P_m(p; \beta)A(p).
\]

\[\text{DRAFT}\]
Then, the sought sufficient condition is obtained by following the same reasoning that has led to Theorem 1. A result analogous to Theorem 2 can also be derived.

IV. EXAMPLES

In this section, some numerical examples are presented to illustrate the proposed technique for robust stability analysis of polytopic systems.

A. Example 1

The first example is deliberately simple, in order to show how the LMIs involved in the sufficient condition are generated. Consider the problem of computing the robust parametric margin \( \rho \) defined as

\[
\rho = \sup \{ \tilde{\eta} \in \mathbb{R} : \ A(p, \eta) \text{ is Hurwitz for all } p \in P \text{ for all } \eta \in [0, \bar{\eta}] \} \tag{29}
\]

where

\[
A(p; \eta) = \sum_{i=1}^{q} p_i A_i(\eta), \quad A_i(\eta) = \tilde{A}_0 + \eta \tilde{A}_i, \quad i = 1, \ldots, q
\]

and \( \tilde{A}_i, i = 0, \ldots, v \) are given matrices with \( \tilde{A}_0 \) Hurwitz. Note that the solution of the above problem amounts to solving a one-parameter family of robust stability problems addressed in the paper, namely one for each fixed value of \( \eta \).

Let us study the numerical example with \( q = 2, n = 2 \) and

\[
\tilde{A}_0 = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}.
\]

Consider first the case \( m = 0 \), which means that a common Lyapunov function is sought for all the matrices of the polytope \( \mathcal{A} \) (in other words, the Lyapunov function does not depend on the uncertain parameter). The sufficient condition in Theorem 1 involves the matrices

\[
S_0(\beta) = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_3 \end{bmatrix}, \quad U_1(\alpha) = \begin{bmatrix} 0 & 0 & 0 & -\alpha_1 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ -\alpha_1 & 0 & 0 & 0 \end{bmatrix},
\]
The number of free parameters is equal to \( u(2, 2, 1) + s(2, 2, 0) = 1 + 3 = 4 \). By solving the LMI (24) for different values of \( \eta \), one gets the lower bound \( \hat{\rho}_{m=0} = 0.321 \).

Consider now the case \( m = 1 \), which corresponds to choosing a linearly parameter-dependent quadratic Lyapunov function. In this case, the matrices involved in (24) are

\[
\bar{S}_1(\beta) = \begin{bmatrix}
\beta_1 & \beta_2 & 0 & -\beta_7 \\
\beta_2 & \beta_3 & \beta_7 & 0 \\
0 & \beta_7 & \beta_4 & \beta_5 \\
-\beta_7 & 0 & \beta_5 & \beta_6 \\
\end{bmatrix}, \quad \bar{U}_2(\alpha) = \begin{bmatrix}
0 & 0 & 0 & -\alpha_1 & \alpha_3 & -\alpha_2 & -\alpha_4 \\
0 & 0 & \alpha_1 & 0 & \alpha_2 & -\alpha_5 \\
0 & \alpha_1 & 2\alpha_3 & \alpha_4 & 0 & -\alpha_6 \\
-\alpha_1 & 0 & \alpha_4 & 2\alpha_5 & \alpha_6 & 0 \\
-\alpha_3 & \alpha_2 & 0 & \alpha_6 & 0 & 0 \\
-\alpha_2 - \alpha_4 & -\alpha_5 & -\alpha_6 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
\bar{R}_2(\beta) = \begin{bmatrix}
r_1 & r_2 & 0 & 0 & 0 & 0 \\
r_2 & r_3 & 0 & 0 & 0 & 0 \\
0 & 0 & r_7 & r_8 & 0 & 0 \\
0 & 0 & r_8 & r_9 & 0 & 0 \\
0 & 0 & 0 & 0 & r_4 & r_5 \\
0 & 0 & 0 & 0 & r_5 & r_6 \\
\end{bmatrix}, \quad r_1 = 2\eta\beta_1 + 4\beta_2, \quad r_2 = -\beta_1 - 2(2\eta - 1)\beta_2 + 2\beta_3, \quad r_3 = -2\beta_2 - 2(5\eta - 2)\beta_3, \quad r_4 = -2\eta\beta_1 + 4\beta_2, \quad r_5 = -\beta_1 + 2(2\eta + 1)\beta_2 + 2\beta_3, \quad r_6 = 2 - \beta_2 + 2(5\eta + 2)\beta_3.
\]

The number of free parameters is increased to \( u(2, 2, 2) + s(2, 2, 1) = 6 + 7 = 13 \). The LMI (24) turn out to be feasible for all values of \( \eta \) up to \( \hat{\rho}_{m=1} = 0.463 \). This is also the value of the true robust parametric margin (i.e., \( \rho = \hat{\rho}_{m=1} \)), because for \( p = [0 \ 1]' \) and \( \eta = \sqrt{\frac{\pi - 1}{5}} \approx 0.463 \) the matrix \( A(p; \eta) \) has an eigenvalue equal to zero. This means that for this example condition (24) with \( m = 1 \) is also necessary. This example confirms the potential of parameter-dependent Lyapunov function, which is well-known in robust control literature.
B. Example 2

Consider the problem of computing the robust parametric margin $\rho$ defined in (29)-(30), with $q = 2$, $n = 3$ and

$$\bar{A}_0 = \begin{bmatrix} -2 & 1 & -1 \\ 2.5 & -3 & 0.5 \\ -1 & 1 & -3.5 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} -0.7 & -0.5 & -2 \\ -0.8 & 0 & 0 \\ 1.5 & 2 & 2.4 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 0.7 & 0.5 & 2 \\ 0.8 & 0 & 0 \\ -1.5 & -2 & -2.4 \end{bmatrix}.$$  

From Theorem 1, one obtains the following lower bounds:

$\hat{\rho}_{\{m=0\}} = 1.676$; $\hat{\rho}_{\{m=1\}} = 3.208$; $\hat{\rho}_{\{m=2\}} = \rho = 3.552$. The number of free parameters in the LMIs is equal to 9, 30 and 69, respectively. It can be observed that in this example, the true robust parametric margin is achieved for $m = 2$, i.e. by using a HDP-QLF in which the dependence on the uncertain parameter is quadratic. The family of linearly parameter-dependent QLFs ($m = 1$) is conservative in this case.

Finally, the technique proposed in [15], that provides a sufficient condition for robust stability of a polytopic system which includes the sufficient conditions proposed in [10], [13], [14], has been applied to this example obtaining the lower bound $\hat{\rho}_L = 3.208$ which is equal to $\hat{\rho}_{\{m=1\}}$.

C. Example 3

Let us consider another problem similar to those in the previous examples, with $q = 3$, $n = 4$ and

$$\bar{A}_0 = \begin{bmatrix} -2.4 & -0.6 & -1.7 & 3.1 \\ 0.7 & -2.1 & -2.6 & -3.6 \\ 0.5 & 2.4 & -5 & -1.6 \\ -0.6 & 2.9 & -2 & -0.6 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} -0.7 & -0.5 & -2 \\ -0.8 & 0 & 0 \\ 1.5 & 2 & 2.4 \\ 0.7 & 0.5 & 2 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} 1.1 & -0.6 & -0.3 & -0.1 \\ -0.8 & 0.2 & -1.1 & 2.8 \\ -1.9 & 0.8 & -1.1 & 2 \\ -2.4 & -3.1 & -3.7 & -0.1 \end{bmatrix},$$

$$\bar{A}_2 = \begin{bmatrix} 0.9 & 3.4 & 1.7 & 1.5 \\ -3.4 & -1.4 & 1.3 & 1.4 \\ 1.1 & 2 & -1.5 & -3.4 \\ -0.4 & 0.5 & 2.3 & 1.5 \end{bmatrix}, \quad \bar{A}_3 = \begin{bmatrix} -1 & -1.4 & -0.7 & -0.7 \\ 2.1 & 0.6 & -0.1 & -2.1 \\ 0.4 & -1.4 & 1.3 & 0.7 \\ 1.5 & 0.9 & 0.4 & -0.5 \end{bmatrix}.$$  

From Theorem 1, one obtains the following lower bounds:

$\hat{\rho}_{\{m=0\}} = 1.019$; $\hat{\rho}_{\{m=1\}} = 1.968$; $\hat{\rho}_{\{m=2\}} = \rho = 2.224$. The number of free parameters in the LMIs is equal to 28, 198 and 750, respectively.
By using the approach proposed in [15], one gets the lower bound \( \hat{\rho}_L = 1.8784 \). Note that in this case \( \hat{\rho}_L < \hat{\rho}_{(m=1)} \), which suggests that the results provided by HDP-QLF with \( m = 1 \) can be less conservative than those obtained via existing techniques employing linearly parameter-dependent Lyapunov functions.

V. CONCLUSIONS

Sufficient conditions for robust stability of linear systems affected by polytopic uncertainty have been provided. The novelty of the approach lies in the use of Lyapunov functions whose dependence on the uncertain parameters is expressed as a homogeneous polynomial form. The obtained conditions are formulated in terms of LMI feasibility tests.

Several interesting developments can be foreseen. Necessity of the proposed conditions is related to the possibility of representing positive definite matrices of homogenous forms as the sum of squares of suitable matricial forms. Studying this representation would lead to a deeper understanding of the conservativeness degree of the sufficient condition. Comparisons with linearly parameter-dependent Lyapunov functions must be pursued in order to clarify if the adopted parameterization of matricial forms encompasses the linear parameterizations proposed in the literature. The use of homogeneous parameter-dependent Lyapunov functions for the synthesis of robust controllers for polytopic systems is also under investigation.

REFERENCES


