Part 3.3 Trellis Coded Modulation
Overview of TCM (1)

- Conventional coding

  - Separate from modulation, performed at the digital level before modulation

  - *The insertion of redundant bits*

    - Given the same information transmission rate, the symbol rate must be \((n/k)\) times that of the uncoded system.
    
    - The redundancy provides coding gain, however, *requires extra bandwidth*.

  - *In a band-limited channel*, the required additional bandwidth is unavailable.
Overview of TCM (2)

➢ **Solution:** *Trellis coded modulation (TCM)*

- The combination of coding and modulation
- Coding gain without expanding bandwidth
  - *Using a constellation with more points* than that required without coding
  - Typically, the number of points is doubled
  - The symbol rate is unchanged and *the bandwidth remains unchanged*. 
Overview of TCM (3)

➢ How to achieve the coding gain by TCM?

- **Introducing dependancy between every successive symbols**
  - Only certain sequences of successive constellation points are allowed

- **Maximizing the Euclidean distance between possible sequences of transmitted symbols**
  - Minimum distance between the possible sequences of transmitted symbols in signal space ($d_{min}$) determines the performance:
    \[ P_e \propto e^{-\frac{d_{min}^2}{\sigma_n^2}} \]
  - It actually decreases the error probability for a given SNR, thus achieving coding gain
History of TCM

Ungerboeck Invented TCM 1976

Rotationally Invariant TCM 1983
  • Voiceband Modems up to 14.4 Kbps

Multidimensional TCM 1984 - 1985
  • Voiceband Modems up to 33.4 Kbps

Rotationally Invariant TCM with M-PSK 1988
  • Satellite Communications

TCM with Built-In Time Diversity 1988 - 1990
  • Wireless Communications Trials

TCM with Tomlinson Precoder 1990 - 1991
  • Digital Subscriber Loops
    • HDTV

TCM with Unequal Error Protection 1990
  • Broadcast Channels

Multilevel Coding with TCM 1992 - 1993
  • Satellite Communications
    • HDTV
    • CATV
    • DBS
    • Digital Subscriber Loops

Concatenated Coding with TCM 1993 - present
Basic Principles of TCM (1)

- **TCM is to devise an effective method for mapping the coded bits into signal points such that the minimum Euclidean distance is maximized.**

- **Ungerboek idea: mapping by set partitioning**
  - The signal constellation is partitioned in a systematic manner to form a series of smaller subsets.
  - The resulting subsets have a larger minimum distance than their “parent”.
  - *The goal of partitioning: each partition should produce subsets with increased minimum distance.*
Example of Set Partitioning

16-QAM $A = 16$ QAM

$B_0$ and $B_1$ are the root nodes.

$C_0$, $C_1$, $C_2$, $C_3$ are the child nodes.

$D_0$, $D_1$, $D_2$, $D_3$, $D_4$, $D_5$, $D_6$, $D_7$ are the leaf nodes.

0000, 1000, 0100, 1100, 0010, 1010, 0110, 1110, 0001, 1001, 0101, 1101, 0011, 1011, 0111, 1111
Basic Principles of TCM (2)

- **In general, the encoding is performed as follows:**
  - A block of $m$ information bits is separated into two groups of length $k_1$ and $k_2$, respectively.
  - The $k_1$ bits are encoded into $n$ bits, while the $k_2$ bits are left uncoded.
  - The $n$ bits from the encoder are used to select one of the possible subsets in the partitioned signal set, while the $k_2$ bits are used to select one of $2^{k_2}$ signal points in each subset.

- The coder need not code all the incoming bits. When $k_2=0$, all $m$ information bits are encoded.

- There are many ways to map the coded bits into symbols. The choice of mapping will drastically affect the performance of the code.
Basic Principles of TCM (3)

- **General structure of encoder:**

```
   1  2  \vdots  k_1
   \vdots  \vdots  \vdots
   \vdots  \vdots  \vdots
   k_2

   Binary encoder

   \rightarrow

   Select subset
   \{1, 2, \ldots, 2^n\}

   \vdots  \vdots  \vdots
   \vdots  \vdots  \vdots
   \vdots  \vdots  \vdots
   n

   Select point from subset
   \{1, 2, \ldots, 2^k_2\}

   \rightarrow

   Signal point

   Uncoded bits
```
Basic Principles of TCM (4)

The basic rules for the assignment of signal subsets to state transitions in the trellis

- Use all subsets with equal frequency in the trellis

- Transitions originating from the same state or merging into the same state in the trellis are assigned subsets that are separated by the largest Euclidean distance

- Parallel state transitions (when they occur) are assigned signal points separated by the largest Euclidean distance.
  - Parallel transitions in the trellis are characteristic of TCM that contains one or more uncoded information bits.
Examples of TCM (1)

8-PSK constellation partition

\[ \text{distance} = \sqrt{2} \varepsilon \]
Examples of TCM (2)

(a) Encoder

(b) Four-state trellis

Parallel transition

(c) Mapping of bits to state transitions

(d) Mapping of bits \((c_3, c_2, c_1)\) to signal points corresponding to partition in Fig. 8.3-1 (note nonuniqueness of this mapping)

\(\frac{1}{2}\) convolutional encoder with 4 states
Examples of TCM (3)

Minimum Euclidean distance:

\[ d_{\text{min, uncoded}} = \sqrt{2\epsilon} \]

Uncoded QPSK

Trellis coded 8PSK modulation

Distance: \((0, 0, 0) \rightarrow (2, 1, 2)\)

\[ d^2_{000 \rightarrow 212} = d_0^2 + 2d_1^2 \]

\[ = \left[ (2 \cdot \sqrt{2})\epsilon + 4\epsilon \right] = 4.585\epsilon \]

Parallel transition:

Distance: \((0, 0, 0) \rightarrow (4, 0, 0)\)

\[ d^2_{000 \rightarrow 400} = d_2^2 = 4\epsilon \]
Examples of TCM (4)

2/3 TCM encoder with 8 states

$k = 2$ bits

$c_1, c_2, c_3$
Examples of TCM (5)

No parallel transition

Distance : $$(0, 0, 0) \rightarrow (6, 7, 6)$$

$$d_{000 \rightarrow 676}^2 = d_0^2 + 2d_1^2$$

$$= \left[ (2 - \sqrt{2})\varepsilon + 4\varepsilon \right] = 4.585\varepsilon$$
Coding Gain (1)

- The minimum Euclidean distance between paths that diverge from any state and remerge at the same state in the trellis code is called **free Euclidean distance** $D_{\text{fed}}$.

- Asymptotic coding gain:

$$\gamma = \left( \frac{E_{\text{uncoded}}}{d_{\text{min,uncoded}}^2} \right) \left/ \left( \frac{E_{\text{coded}}}{D_{\text{fed,coded}}^2} \right) \right.$$  

where $E$ is the normalized average received energy.

When $E_{\text{uncoded}} = E_{\text{coded}}$,  
$$\gamma = \frac{D_{\text{fed,coded}}^2}{d_{\text{min,uncoded}}^2}$$

In the 4-state example,  
$$D_{\text{fed}} = 2\sqrt{\epsilon}, \ d_{\text{min,uncoded}} = \sqrt{2}\epsilon$$  

$\gamma = 2 \Rightarrow 3\text{dB coding gain}$
Asymptotic coding gain can be increased by increasing the number of states and the rate of the convolutional encoder.

In the 8-state example,

\[ D_{fed} = \sqrt{4.585}\varepsilon, \quad d_{\text{min, uncoded}} = \sqrt{2}\varepsilon \]

\( \gamma = 2.2925 \Rightarrow 3.6\text{dB coding gain} \)
## Coding Gain (3)

<table>
<thead>
<tr>
<th>Number of states</th>
<th>Code rate $k_1 / (k_1 + 1)$</th>
<th>$m = 3$ coding gain (dB) of 16-PSK versus uncoded 8-PSK</th>
<th>$m \rightarrow \infty$ $N_{fed}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>3.54</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>4.01</td>
<td>4</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td>4.44</td>
<td>8</td>
</tr>
<tr>
<td>32</td>
<td>1</td>
<td>5.13</td>
<td>8</td>
</tr>
<tr>
<td>64</td>
<td>1</td>
<td>5.33</td>
<td>2</td>
</tr>
<tr>
<td>128</td>
<td>1</td>
<td>5.33</td>
<td>2</td>
</tr>
<tr>
<td>256</td>
<td>2</td>
<td>5.51</td>
<td>8</td>
</tr>
</tbody>
</table>

Viterbi Decoding (1)

➢ Two steps:

☑ Step 1: At each branch in the trellis,

- Compare the received signal to each of the signals allowed for that branch.
- Save the signal closest to the received signal
- Label the branch with a metric proportional to the Euclidean distance between the two signals.
  
  ➢ Branch metric calculation

Determining the best signal with the smallest distance to the received signal within each subset ➜ subset decoding
Viterbi Decoding (2)

➢ **Two steps:**

➢ **Step 2:**

- Apply the Viterbi algorithm to the trellis, with surviving partial paths corresponding to partial signal sequences that are closest to the received sequences.

- Select the ML path (the complete signal sequence closest in Euclidean distance to the received sequence) at the end of the trellis.

- **Path metric calculation**

- **Trellis update**
Error Rate Performance

- An error event happens when an erroneous path is selected at the decoder

- Error-event probability in AWGN channel:

\[ P_e \approx N_{fed} Q \left( \sqrt{\frac{D_{fed}^2}{2N_0}} \right) \text{ under high SNR} \]

\( N_{fed} \rightarrow \) the number of signal sequences with distance \( D_{fed} \) that diverge at any state and remerge at that state after one or more transitions