Behavioral Descriptions of Non-linear Electrical Dynamic Systems in Qualitative Phase Space

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ABSTRACT

This correspondence describes techniques for analyzing the overall behaviors of electrical dynamic systems in qualitative phase space. It is to combine the existing methodologies that describe non-linear dynamic systems in phase space and the descriptions of qualitative behaviors of ordinary differential equations. The Van der Pol equation has been illustrated as an example.

KEYWORDS: qualitative reasoning, phase plane analysis, Van der Pol equation.

I INTRODUCTION

It is important to consider what qualitative information can be obtained about these systems without actually solving the equations. In nature, most dynamic systems are non-linear, and analytical methods cannot be utilized and they will not obey the superposition principle [9]. Sometimes, traditional techniques for analyzing or modeling are impossible since it is the lack of sufficient quantitative information. On the other hand, the qualitative analysis can still provide results for the possible outcomes of the given qualitative information. For the current technology, engineering design process is using complex computational tools [2] and computers to

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numerically analyze and model the systems. Nevertheless, we still need to have the qualitative understanding of the phenomena of the systems to verify those outcomes whether they make sense. Therefore, we need a new method to help us to analyze and model non-linear systems in a qualitative way.

The existing considerations on the case studies are divided into only a subset of the overall system behaviors. If there are different operating regions, the system dynamics can have different behaviors. To consider the different operating regions, we can use the phase space method to specify the interested operating region, which can be partitioned into several sub-regions where the trajectories have common boundary conditions. The phase portrait of a second order differentiation equation can be plotted by transforming the equation into two first order differential equations in Cartesian form. Then, we can interpret and analyze the behaviors of the system by the parameters in the equations.

### II THEORY

**A. Qualitative Reasoning and the QSIM Representation**

Qualitative reasoning is an intuitive method to describe the physical world by incomplete knowledge [1]. The framework of the relationship between differential equations and qualitative representations is shown in Fig. 1 [1]. All the variables in the qualitative reasoning are expressed in both magnitudes and directions. The parameters of a model are continuous differentiable functions in time domain. Each of them uses an ordered set of landmark values to illustrate the quantity spaces in discrete regions, such that the parameters have the same characteristics within that region. The directions of the parameters are part of the qualitative variables. Models can be in different operating regions, which can have their own constraints and

![Fig. 2: The common fixed points](image-url)
qualitative variables. QSIM algorithm [1] reasons those physical systems as autonomous by the qualitative differential equations (QDE). QSIM uses well-defined qualitative constraints to describe the system of QDEs, which predicts the possible behaviors of a system. In the QSIM, the ordered landmark values can be asserted for different variables and can be created during the process while there is a new meaningful critical value found in the interval.

B. Phase Space Analysis

This is a traditional method to illustrate the behavior of a second order differential equation. It is the Cartesian plot for the variable versus its derivative. The trajectories vary with time from the starting points, which is dependent on the initial conditions. For those differential equations that cannot be solved analytically, the behaviors can still be illustrated in the phase portrait. The behaviors can be linearized by Jacobian matrix [4], while near the fixed points (singular points). Fig. 2 shows some examples for fixed points.

C. Van der Pol Equation

This is a model (1) developed to describe the operation of an electronic valve oscillator. This was firstly investigated while electrical circuits employing vacuum tubes. It is because there is only one unstable equilibrium point at the origin. If there is any disturbance (some the initial values are not zero) in the input source, the system starts to oscillate and reach a limit cycle after a few cycles. The limit cycle is governed by a constant $\mu$. There are one harmonic component and one damping term. When $\mu = 0$, it becomes a simple harmonic system (only harmonic component).

$$x''-\mu(1-x^2)x'+x=0 \quad (1)$$

where $\mu$ is a non-zero constant. Obviously, it is a second order nonlinear dynamic

![Fig. 3: The numerical simulation of Van der Pol Equation ($\mu=0.5$, and $x_1=0.1$, $x_2=0$)](image-url)
system. The phase portrait of Van der Pol can be plotted by transforming the equation into two first order differential equations in Cartesian form [2],[7]. Then, we can interpret and analyze the behaviors of the system by the parameters in the equations. However, there is difficulty to use the piece-wise linearization method to analyze these equations since the equilibrium point is a center.

III SIMULATIONS

A. Numerical Methods

To analyze non-linear differential equations, numerical methods are very useful since the drastically increasing of the computational power of computer. Those trajectories are plotted in the phase portrait without any pre-analysis. The initial states, however, have to be stated before simulations. To get better and predictable outcome, analyses have to be done before making any initial states or even they can be chosen randomly. Therefore, it is possible that some of the behaviors would be missed.

\[
\begin{align*}
  x_1' &= x_2, \\
  x_2' &= \mu(1-x_1^2)x_2 - x_1 \\
\end{align*}
\]  

(2)

The set of first order system differential equations (2) is rewritten from (1). The variable \( x_1 \) is \( x \) and \( x_2 \) is derivative of \( x \). If analyses have been done before making any initial states, the number of try and errors can be reduced and the chosen initial points produce some trajectories that represent the critical behaviors of the model. The Fig. 3 is a phase portrait by numerical simulations in one set of initial states \((\mu=0.5, x_1=0.1 \text{ and } x_2=0)\). The trajectory is oscillating from the initial state clockwise and outward. After about three cycles, the system has reached the stable limit cycle. The dotted lines are the isoclines when \( \frac{dx_2}{dx} = 0 \). The isocline for \( \frac{dx_1}{dx} = 0 \) is horizontal \( x_1 \) axis only. Plotting different \( \mu \) and initial values of \( x_1 \) and \( x_2 \) can help to explore the characteristics of the system.
B. Qualitative Phase Portrait

Qualitative phase portrait [10] is based on qualitative simulation [1]. The qualitative phase portrait is illustrated directly in the phase space without the time domain analysis. The relationship of variables would be examined such that all the possible behaviors are found from the qualitative confluence [3]. The consideration of the confluence applied to partition the qualitative phase space into sub-regions, where the directions of qualitative variables would be same in the sub-regions. Van der Pol equation is quite different from the simplified swing equation used as an example in [10].

In the simple qualitative phase space Fig. 4, there are two axes that partition the whole space into four regions already. The quantity spaces of \([X_1]\) and \([X_2]\) are in the set \({[-], [0], [+]}\). Therefore, nine (3×3) basic states locate in the combination of the two variables, \([X_1]\) and \([X_2]\) in each phase portrait. The set of confluence (3) is rewritten from (2). \(\mu\) is a non-zero positive constant. The direction of \(\partial X_1\) is changing with the qualitative magnitude of \(X_2\) while the \(X_2 \in \{[-], [0], [+]}\). U-(convex upward) constraints [1] are used to describe the damping term (the first part at right hand side in Eq.(3)) in equation.

\[
\begin{align*}
\partial X_1 &= [X_2] \\
\partial X_2 &= [\mu][1 - X_1^2][X_2] - [X_1]
\end{align*}
\] (3)

The Fig. 5 shows all the null-isoclines in the phase portrait. The circle at the origin is the fixed point and all the arrows are representing the directions of all possible trajectories. It is not hard to understand that all the arrows on the horizontal \(x_1\) axis are only upwards or downwards according to the first equation in Eq. (3) with \(\partial X_1 = [X_2] = 0\). When \(\partial X_2 = 0\), the situation is more complicated. There are many arrows on left-upper corner and the right-lower corner. There are only some possible points for the derivative of \(x_2\) against \(x_1\) being zero but not the whole regions are zero derivative. It should be a line to separate the regions. For different \(\mu\), however, the lines are different. Therefore, there is no any precise boundary exists for this kind of non-linear equations.

![Fig. 6: Simplified phase portrait of Van der Pol equation.](image)
Fig. 6 is the simplified phase portrait for Van der Pol equation. There are only the simple shading included. There are two unknown variables $c_1$ and $c_2$ have been added in the model, $c_1 \in (0,1)$ and $c_2 \in (-1,0)$. To locate the exact values of the $c_1$ and $c_2$, the numerical analysis is really needed. It is, however, not necessary for only the purpose on understanding the behaviors of system. It is easy to understand the behaviors of the system from these figures already. Conclusion can be drawn that the system is stable with the limit cycle, since there are no any divergent region and convergent region [10]. Compared with the Fig. 3, Fig. 5 and 6 can provide more general information about the system without the initial conditions and the value of $\mu$.

IV CONCLUSION

Qualitative analysis with the phase portrait, provides useful information about the characteristics of the system. The overall behaviors of the system are illustrated by some qualitative phase portraits without specifying those critical parameters and initial conditions. The demanding numerical computation has been reduced and all possible characteristics can be deduced from the qualitative phase portraits.

The proposed method reduces the spurious behaviors from qualitative simulation [1]. Using the qualitative phase plane analysis before QSIM, it would be helpful to be understood those behaviors. Van der Pol equation is a good example for partitioning the complex phase space to different sub-regions, compared with the simplified swing equation[10]. It is because there are some unclear boundaries existing in the phase portrait.

REFERENCE