Fourier-Bessel Transform

1 Fourier-Bessel Transform

If a function is separable in polar coordinates, we can write

\[ g(r, \theta) = g_r(r)g_\theta(\theta) \]

Furthermore, if it is circularly symmetric, i.e.,

\[ g(r, \theta) = g_r(r), \]

then we can proceed with the following simplifications.

With the following transformations,

\[
\begin{align*}
  r &= \sqrt{x^2 + y^2} & x &= r \cos \theta \\
  \theta &= \arctan \left( \frac{y}{x} \right) & y &= r \sin \theta \\
  \rho &= \sqrt{f_X^2 + f_Y^2} & f_X &= \rho \cos \phi \\
  \phi &= \arctan \left( \frac{f_Y}{f_X} \right) & f_Y &= \rho \sin \phi
\end{align*}
\]

we have

\[
G_0(\rho, \phi) = \int_0^{2\pi} \int_0^\infty g_r(r) e^{-j2\pi \rho \cos \phi} e^{+j2\pi \rho \sin \phi} r \, dr \, d\theta
\]

\[
= \int_0^\infty \left( \int_0^{2\pi} e^{-j2\pi \rho \cos \phi} d\theta \right) r g_r(r) \, dr
\]

The expression still looks complicated. But using the Bessel function defined in Handout #1,

\[
J_0(a) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ja \cos \phi} d\theta.
\]

Figure 1: Spatial domain and frequency domain.
Note that this expression is independent of \( \phi \). We can substitute it to Equation 1 and obtain

\[
G_0(\rho) = 2\pi \int_0^\infty r g_r(r) J_0(2\pi r \rho) \, dr. \tag{1}
\]

Similarly, the inverse Fourier transform can be expressed as

\[
g_r(r) = 2\pi \int_0^\infty \rho G_0(\rho) J_0(2\pi r \rho) \, d\rho. \tag{2}
\]

Fourier-Bessel transform is also called Hankel transform of zero order. We can use \( \mathcal{B}\{\cdot\} \) to represent the Fourier-Bessel transform operation. It follows that

\[
\mathcal{BB}^{-1}\{g_r(r)\} = \mathcal{B}^{-1}\mathcal{B}\{g_r(r)\} = \mathcal{BB}\mathcal{B}\{g_r(r)\} = g_r(r)
\]

at each value of \( r \) where \( g_r(r) \) is continuous.