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Evaluation of Certain Fourier Transforms

1 Direct integration: Fourier transform of $\sqcap(x)$

The straightforward way of computing Fourier transform is by direct integration. This is appropriate for evaluating $\mathfrak{F}\{\sqcap(x)\}$. Note that it is a real and even function, and we expect its Fourier transform is real and even.

$$\begin{aligned}\mathfrak{F}\{\sqcap(x)\} &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{-\mathrm{j}2\pi s x}\,\mathrm{d}x \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2\pi s x)\,\mathrm{d}x \\ &= \frac{\sin \pi s}{\pi s} \equiv \operatorname{sinc}(s)\end{aligned}$$

The sinc function has the properties that $\operatorname{sinc}(s) = 0$ when s is at integer positions (since $\sin(\pi s) = 0$, except at zero where both the numerator and denominator are zero), $\operatorname{sinc}(0) = 1$ using L'Hôpital's rule, and its envelope falls off with $\frac{1}{\pi s}$.

2 Symmetric relationship: Fourier transform of $\operatorname{sinc}(x)$

Every time we derive a Fourier transform relationship, we actually get them in pairs. This can be seen by noting the following: If $g(x)$ and $G(s)$ are Fourier transform pairs, we can write

$$\begin{aligned}g(x) &= \int_{-\infty}^{\infty} G(s)\mathrm{e}^{\mathrm{j}2\pi s x}\,\mathrm{d}s \\ g(-x) &= \int_{-\infty}^{\infty} G(s)\mathrm{e}^{\mathrm{j}2\pi s(-x)}\,\mathrm{d}s \\ &= \mathfrak{F}\{G(s)\}\end{aligned}$$

Thus, we have

$$\boxed{g(x) \supset G(s) \qquad \Longrightarrow \qquad G(x) \supset g(-s).}$$

Furthermore, if $g(s)$ is an even function, $G(x) \supset g(s)$. This is the case for $\operatorname{sinc}(x)$, whose Fourier transform is $\sqcap(s)$.

3 Using Convolution Relationship: Fourier Transform of $\wedge(x)$

It is possible to use direct integration to compute $\mathfrak{F}\{\wedge(x)\}$, but a easier way is to note that $\wedge(x) = \sqcap(x) * \sqcap(x)$. Since convolution maps to multiplication in the Fourier domain,

$$\begin{aligned}\mathfrak{F}\{\wedge(x)\} &= \mathfrak{F}\{\sqcap(x)\} \cdot \mathfrak{F}\{\sqcap(x)\} \\ &= \operatorname{sinc}^2(s).\end{aligned}$$

4 Transform in the Limit: Fourier Transform of $\mathrm{sgn}(x)$

The signum function is real and odd, and therefore its Fourier transform is imaginary and odd. If we attempt to evaluate the Fourier transform integral directly, we get

$$\begin{aligned}
\mathfrak{F}\{\mathrm{sgn}(x)\} &= \int_{-\infty}^{\infty} \mathrm{sgn}(x) \mathrm{e}^{-\mathrm{j}2\pi sx}\, \mathrm{d}s \\
&= -\mathrm{j} \int_{-\infty}^{\infty} \mathrm{sgn}(x) \sin(2\pi sx)\, \mathrm{d}s \\
&= -2\mathrm{j} \int_{0}^{\infty} \sin(2\pi sx)\, \mathrm{d}s \\
&= 2\mathrm{j} \left[\frac{\cos(2\pi sx)}{2\pi s} \right]_0^{\infty} \\
&= \mathrm{j} \left[\frac{\cos(2\pi sx)}{\pi s} \right]_{x=\infty} - \frac{\mathrm{j}}{\pi s}.
\end{aligned}$$

The first term in the last expression does not exist, because cosine function does not die down to zero (or any value) as its argument approaches infinity. Thus, in the strict sense of Fourier transform, $\mathfrak{F}\{\mathrm{sgn}(x)\}$ does not exist.

This trouble arises because the signum function is not absolutely integrable, i.e. it violates the condition on $f(x)$ that $\int_{-\infty}^{\infty} |f(x)| < \infty$. However, it is possible to evaluate its Fourier transform if we allow the following "extended" method of Fourier transform in the limit:

3 Steps for Fourier transform in the limit:

1. Define a sequence of functions $f_N(x)$ such that $\lim_{N \to \infty} f_N(x) = f(x)$
2. Compute the Fourier transform: $F_N(s) = \mathfrak{F}\{f_N(x)\}$
3. Define $F(s)$ such that $F(s) \equiv \lim_{N \to \infty} F_N(s)$

There can be many possibilities in defining $f_N(x)$. For the signum function, one possibility is to define

$$f_N(x) = \begin{cases} \mathrm{e}^{-\frac{x}{N}} & : \quad x \geq 0 \\ -\mathrm{e}^{\frac{x}{N}} & : \quad x < 0 \end{cases}$$

We can check that as $N \to \infty$, $f_N(x) \to \mathrm{sgn}(x)$. Therefore,

$$\begin{aligned}
F_N(s) &= \int_0^{\infty} \mathrm{e}^{-\frac{x}{N}} \mathrm{e}^{-\mathrm{j}2\pi sx}\, \mathrm{d}x - \int_{-\infty}^{0} \mathrm{e}^{\frac{x}{N}} \mathrm{e}^{-\mathrm{j}2\pi sx}\, \mathrm{d}x \\
&= \int_0^{\infty} \mathrm{e}^{-\left(\frac{1}{N}+\mathrm{j}2\pi s\right)x}\, \mathrm{d}x - \int_0^{\infty} \mathrm{e}^{-\left(\frac{1}{N}-\mathrm{j}2\pi s\right)y}\, \mathrm{d}y \qquad (y = -x) \\
&= \left[\frac{\mathrm{e}^{-\left(\frac{1}{N}+\mathrm{j}2\pi s\right)x}}{-\left(\frac{1}{N}+\mathrm{j}2\pi s\right)} \right]_{x=0}^{\infty} - \left[\frac{\mathrm{e}^{-\left(\frac{1}{N}-\mathrm{j}2\pi s\right)x}}{-\left(\frac{1}{N}-\mathrm{j}2\pi s\right)} \right]_{x=0}^{\infty} \\
&= \frac{1}{\frac{1}{N}+\mathrm{j}2\pi s} - \frac{1}{\frac{1}{N}-\mathrm{j}2\pi s}
\end{aligned}$$

We can then define the Fourier transform of the signum function as

$$
\begin{aligned}
\mathfrak{F}\{\mathrm{sgn}(x)\} &\equiv \lim_{N\to\infty}\left(\frac{1}{\frac{1}{N}+\mathrm{j}2\pi s}-\frac{1}{\frac{1}{N}-\mathrm{j}2\pi s}\right) \\
&= \frac{1}{\mathrm{j}2\pi s}+\frac{1}{\mathrm{j}2\pi s} \\
&= -\frac{\mathrm{j}}{\pi s}
\end{aligned}
$$

Using the symmetric relationship of the two domains discussed above in Section 2, we can readily see that

$$
\frac{1}{x} \supset -\mathrm{j}\pi\mathrm{sgn}(s)
$$

5 Fourier Transform of the Gaussian function $\mathrm{e}^{-\pi x^2}$

Putting the Gaussian function in the Fourier transform equation, we get

$$
\begin{aligned}
\mathfrak{F}\{\mathrm{e}^{-\pi x^2}\} &= \int_{-\infty}^{\infty} \mathrm{e}^{-\pi x^2}\mathrm{e}^{-\mathrm{j}2\pi sx}\ \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-\pi(x^2+\mathrm{j}2sx+\mathrm{j}^2s^2)}\mathrm{e}^{-\pi s^2}\ \mathrm{d}x \\
&= \mathrm{e}^{-\pi s^2}\int_{-\infty}^{\infty} \mathrm{e}^{-\pi(x+\mathrm{j}s)^2}\ \mathrm{d}x \\
&= \mathrm{e}^{-\pi s^2}\int_{-\infty}^{\infty} \mathrm{e}^{-\pi y^2}\ \mathrm{d}y \qquad (\text{letting } y = x+\mathrm{j}s)
\end{aligned}
$$

We therefore need to evaluate the area under the Gaussian function. A neat trick, reportedly due to Euler, shows that this equals to unity without resorting to integration tables. Let

$$
I = \int_{-\infty}^{\infty} \mathrm{e}^{-\pi y^2}\ \mathrm{d}y.
$$

Because y is a dummy variable above, we could have used x instead. Thus,

$$
\begin{aligned}
I^2 &= \left(\int_{-\infty}^{\infty} \mathrm{e}^{-\pi x^2}\ \mathrm{d}x\right)\left(\int_{-\infty}^{\infty} \mathrm{e}^{-\pi y^2}\ \mathrm{d}y\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\pi(x^2+y^2)}\ \mathrm{d}x\ \mathrm{d}y,
\end{aligned}
$$

since there is no coupling between the two integrals. We can now consider (x, y) as the Cartesian coordinates, and convert the integral to polar coordinates. Therefore,

$$
\begin{aligned}
I^2 &= \int_0^{2\pi}\int_0^{\infty} \mathrm{e}^{-\pi r^2} r\ \mathrm{d}r\ \mathrm{d}\theta \\
&= (2\pi)\left[-\frac{1}{2\pi}\mathrm{e}^{-\pi r^2}\right]_{r=0}^{\infty} \\
&= 1.
\end{aligned}
$$

So $I = 1$, and

$$
\mathrm{e}^{-\pi x^2} \supset \mathrm{e}^{-\pi s^2}.
$$

6 Fourier Transform of impulse function $\delta(x)$

To calculate the Fourier transform of the impulse function, we make use of its sifting property, so that

$$\begin{aligned}\mathfrak{F}\{\delta(x)\} &= \int_{-\infty}^{\infty} \delta(x)\mathrm{e}^{-\mathrm{j}2\pi sx}\,\mathrm{d}s \\ &= \mathrm{e}^{-\mathrm{j}2\pi s(0)} \\ &= 1.\end{aligned}$$

Since $\delta(x) \supset 1$ also implies $1 \supset \delta(s)$, we have the identity

$$\delta(s) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}2\pi sx}\,\mathrm{d}x.$$

7 Fourier Transform of comb function $\mathrm{III}(x)$

Since $\mathrm{III}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$,

$$\mathfrak{F}\{\mathrm{III}(x)\} = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\mathrm{j}2\pi ns} = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{j}2\pi ns}. \tag{1}$$

We may notice that when s takes on integer values, the expression will blow up, while for other values of s, the complex exponentials will cancel each other and the expression will become zero. It is tempting to suggest that this expression therefore equals to $\mathrm{III}(s)$. It is indeed true; but the line of argument presented above only suggests that the result is $a\mathrm{III}(s)$ where a is some constant. We need a more rigorous mathematical analysis to show that $a = 1$.

To do that, we need to make use of Fourier Series decomposition. We are at liberty to use this tool, because although it is closely related to Fourier transform, its derivation does not involve the use of comb function, so we are not making any circular argument here. Using Fourier series, every periodic function $F(s)$ with period L can be expressed as

$$F(s) = \sum_{n=-\infty}^{\infty} \alpha_n \mathrm{e}^{\mathrm{j}2\pi \frac{n}{L} s} \qquad \text{where} \qquad \alpha_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} F(s)\mathrm{e}^{-\mathrm{j}2\pi \frac{n}{L} s}\,\mathrm{d}s. \tag{2}$$

Note that $\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{j}2\pi ns}$ is periodic in s with period $L = 1$, and that comparing Equation 1 and Equation 2 we can see that $\alpha_n = 1$.

Now letting $G(s) = \sqcap(s)F(s)$ produces one period of the function $F(s)$, and we can write $F(s) = \sum_{n=-\infty}^{\infty} G(s-n)$. To find $G(s)$, we make use of the formula for α_n above, i.e.

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} F(s)\mathrm{e}^{-\mathrm{j}2\pi ns}\,\mathrm{d}s &= 1 \\ \int_{-\infty}^{\infty} G(s)\mathrm{e}^{-\mathrm{j}2\pi ns}\,\mathrm{d}s &= 1 \\ G(s) &= \delta(s).\end{aligned}$$

Therefore,

$$F(s) = \sum_{n=-\infty}^{\infty} \delta(s-n) = \mathrm{III}(s).$$

8 Fourier Transform of trigonometric functions $\sin(x)$ and $\cos(x)$

We can express the trigonometric functions in terms of complex exponentials, where

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \qquad \text{and} \qquad \sin(x) = \frac{e^{jx} - e^{-jx}}{2j}.$$

Since $\mathfrak{F}\{e^{jx}\} = \int_{-\infty}^{\infty} e^{-j2\pi(s - \frac{1}{2\pi})x}\, dx = \delta\left(s - \frac{1}{2\pi}\right)$, and similarly $e^{-jx} \supset \delta\left(s + \frac{1}{2\pi}\right)$, we have

$$\begin{aligned}
\cos(x) &\supset \frac{1}{2}\left[\delta\left(s + \frac{1}{2\pi}\right) + \delta\left(s - \frac{1}{2\pi}\right)\right] \\
\sin(x) &\supset \frac{j}{2}\left[\delta\left(s + \frac{1}{2\pi}\right) - \delta\left(s - \frac{1}{2\pi}\right)\right].
\end{aligned}$$

It is useful to remember that cosine is real and even, and therefore its transform is also real and even, while sine is real and odd, and therefore its transform is imaginary and odd.

Another useful version to remember is the Fourier transform of $\cos(2\pi ax)$ and $\sin(2\pi ax)$. Using the results above, we have

$$\begin{aligned}
\cos(2\pi ax) &\supset \frac{1}{2}\left[\delta\left(s + a\right) + \delta\left(s - a\right)\right] \\
\sin(2\pi ax) &\supset \frac{j}{2}\left[\delta\left(s + a\right) - \delta\left(s - a\right)\right].
\end{aligned}$$