Fourier Transform in Electromagnetics

In electromagnetics, the propagation behaviors of electromagnetic waves were studied and analyzed by plane waves, which was generally more tractable analytically. However, a plane wave is an idealization that does not exist in the real world. In practice, waves are nonplanar in nature as they are generated by finite sources, such as antennas and scatterers. Fortunately, these waves can be expanded in terms of plane waves by using Fourier transform. Once this is done, then the study of non-plane-wave propagation becomes routine.

Take the point source as an example, the spectral decomposition or the plane-wave expansion of the field due to a point source could be derived in a manner described as the following.

According to the electromagnetic theory, the scalar wave equation with a point source is

\[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_0^2 \phi(x, y, z) = -\delta(x)\delta(y)\delta(z) \]  

(1)

The above equation could then be solved in the spherical coordinates, yielding the solution

\[ \phi(r) = \frac{e^{ik_0 r}}{4\pi r} \]  

(2)

Through the Fourier transform, we obtain

\[ \phi(x, y, z) = \frac{1}{(2\pi)^3} \int \int \int d_k x d_k y d_k z \Phi(k_x, k_y, k_z) e^{ik_x x + ik_y y + ik_z z} \]  

(3)

Then, using the above, together with the Fourier representation of the delta function, we convert (1) into

\[ \int \int \int d_k x d_k y d_k z \left[ k_0^2 - k_x^2 - k_y^2 - k_z^2 \right] \Phi(x, y, z) e^{ik_x x + ik_y y + ik_z z} \]

\[ = -\int \int \int d_k x d_k y d_k z e^{ik_x x + ik_y y + ik_z z} \]

(4)

Since the above is equal for all x, y, and z, we must have

\[ \Phi(x, y, z) = \frac{1}{k_0^2 - k_x^2 - k_y^2 - k_z^2} \]

(5)

Consequently, we have

\[ \phi(x, y, z) = \frac{-1}{(2\pi)^3} \int \int \int d_K \left[ k_0^2 - k_x^2 - k_y^2 - k_z^2 \right] e^{ik_x x + ik_y y + ik_z z} = \frac{e^{ik_0 r}}{4\pi r} \]

(6)

Eventually, we can get the Sommerfeld identity by using Bessel functions

\[ e^{ik_0 r} = i \int_{-\infty}^{+\infty} dk_\rho \frac{k_\rho}{k_z} J_0(k_\rho) e^{ik_\rho |z|} \]

(7)

Its physical interpretation is that a spherical wave can be expanded as an integral summation of conical waves or cylindrical waves in the \( \rho \) direction, times a plane wave in the z direction over all wavenumbers \( k_\rho \). This wave is evanescent in the ±z direction when \( k_\rho > k_0 \). This identity is extremely useful in Electromagnetics!