Fourier Transform in Image Processing

2-D Fourier Transform is defined as

$$F(u, v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)e^{-j2\pi(xu+yu)}dx\,dy$$

Perhaps the most common application of 2-D Fourier Transform is in the area of image processing, in which the element $a_{mn}$ of the matrix $A$ represents the pixel value of the .bmp image at the place $(m, n)$. Or, we can obtain the matrix $A$ by sampling the image. Let $I(x, y)$ represents pixel value, then we have:

$$a_{mn} = I((m-1)t_x, (n-1)t_y)$$

where $m = 1, 2 \ldots M$ and $n = 1, 2 \ldots N$, $t_x$ and $t_y$ represent the sampling intervals at x and y axels, respectively.

After the 2-D fourier transform of the image is generated, we can change the properties of the image by manipulating its frequency domain data. One of these manipulations is filtering. The low-pass and high-pass ideal filter is defined as below:

$$H_L(s) = \begin{cases} 
1 & \text{if } |s| \leq s_0 \\
0 & \text{if } |s| > s_0 
\end{cases}$$

$$H_H(s) = \begin{cases} 
0 & \text{if } |s| \leq s_0 \\
1 & \text{if } |s| > s_0 
\end{cases}$$

where $s_0$ is called cut-off frequency.

According to the properties of 2-D fourier transform, we have

$$\frac{\partial I(x, y)}{\partial x} + \frac{\partial I(x, y)}{\partial y} \leftrightarrow H_1(u, v)F(u, v) \quad \frac{\partial^2 I(x, y)}{\partial x\partial y} \leftrightarrow H_2(u, v)F(u, v)$$

where $H_1(u, v) = j2\pi(u + v)$ and $H_2(u, v) = -uv$.

After processing the image in frequency domain, we can perform inverse fourier transform `ifft2` on $F$ and then export a new BMP image.

Here are some images I processed. The original picture is as Fig 1. Fig 2 is the picture after low-pass filtering. Fig 3 is the picture after high-pass filtering. Fig 4 is the picture after low-pass filtering of the green pixel of the original picture. Fig 5 is the picture after multiplying with $H_1(u, v)$ in the frequency domain. Fig 6 is the picture after multiplying $H_2(u, v)$ in the frequency domain.